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Chattering analysis in sliding mode systems with inertial sensors

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Singularly perturbed relay control systems with second order sliding modes (SP2SM) are considered for modeling of sliding mode control systems with inertial sensors. It is shown that the asymptotically stable slow-motion integral manifold of a smooth singularly perturbed system, describing the motion of an original SP2SM in the second order sliding domain, is the asymptotically stable slow-motions integral manifold of the original SP2SM. For sliding mode control systems with inertial sensors sufficient conditions for the exponential decreasing of the amplitude of chattering and unlimited growth of frequency are found. A formula for asymptotic representation of 'ideal' switching surface oscillations is suggested for sliding mode systems with inertial sensors.

1. Introduction

The chattering phenomenon is one of the current problem in modern sliding mode control theory. The presence of inertial sensors is one of the basic reasons for chattering in sliding mode control systems (Bondarev *et al*. 1985, Utkin 1992). In sliding mode systems with inertial sensors the relay control is switched on the surface which is designed based on input variables of the sensor, and the *continuous* plant output is transmitted to the input of the sensor. Fridman and Levant (1996) showed that the behaviour of sliding mode systems with actuators and inertial sensors is described by the relay control systems with higher order sliding modes and the order of sliding is the sum of relative degrees of plant, actuator and sensor.

Bartolini *et al*. (1998) suggested suboptimal control algorithm ensuring the finite time convergence to the second order sliding domain for chattering elimination in systems with actuators with relative degree one. Shkolnikov and Shtessel (2002) have implemented the second order sliding mode control algorithms on dynamics sliding manifolds eliminating the chattering in systems with uncertain actuators.

Sufficient conditions for stability investigation for sliding mode control systems with *fast* actuators of relative degree 1 were found by Fridman (1990). The chattering phenomenon for sliding mode control systems with *fast* actuators, whose behaviour is described by singularly perturbed relay control systems with the order of sliding three and more, was analysed by Fridman (2001) from the view point of averaging.

This paper is devoted to the chattering phenomenon in sliding mode control systems with *fast inertial sensors* given by the singularly perturbed relay control systems with second order sliding modes (SP2SM). The main specific features of relay systems with the second order sliding modes are the following:

- the second order sliding modes could be stable,
- the time of input convergence to the second order sliding mode is infinite, the number of switchings is infinite too (Anosov 1959, Fridman and Levant 1996) and for such systems the classical methods of singular perturbations based on the separation of spectra are not useful.

These specific features of SP2SM determined the main goals of the paper:

- to show that it is possible to design chattering free sliding mode control systems with inertial sensors in the case when the order of sliding in the complete model of the system is 2;
- to obtain the algorithms which take into account the presence of inertial sensors in sliding mode control systems;
- to design mathematical tools for the investigation of SP2SM.

The paper has the following structure. In § 2 the following mathematical tools for investigation of SP2SM are developed:

- sufficient conditions for the exponential decreasing of the amplitude of chattering and unlimited growth of frequency are found;
- it is shown that the asymptotically stable slow-motions integral manifold of smooth singularly perturbed system, describing the motion of the original SP2SM in the second order sliding domain, is the asymptotically stable slow-motions integral manifold of the original SP2SM;
- the reduction principle theorem is proved ensuring the equivalence for stability investigating problems for systems the original SP2SM and the

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reduced system describing dynamics in the slow-motions manifold.

The properties of chattering in sliding mode systems with fast inertial sensors are studied in § 3. The algorithm for correcting the sliding mode equations is suggested in § 3.3 taking into account the presence of fast inertial sensors. A formula for asymptotic representation of 'ideal' switching surface oscillations is given § 3.4. In § 3.5 it is shown that whenever the sliding motions of the plant are stable, but not asymptotically stable, *it is obligatory to make a correction* to the sliding mode equations taking into account the presence of sensors in the system.

2. Decomposition theorem

2.1. *Problem formulation*

Consider the system of the form

$$\left.\begin{aligned} \mu\, \mathrm{d}\bar\eta/\mathrm{d}t &= \bar B_{11}\bar\eta + \bar B_{12}\sigma + \bar B_{13}\xi + \bar B_{14}x \\ \mu\, \mathrm{d}\sigma/\mathrm{d}t &= \bar B_{21}\bar\eta + \bar B_{22}\sigma + \bar B_{23}\xi + \bar B_{24}x + bU(x, u(\xi)) \\ \mu\, \mathrm{d}\xi/\mathrm{d}t &= \sigma \\ \mathrm{d}x/\mathrm{d}t &= \bar B_{41}\bar\eta + \bar B_{42}\sigma + \bar B_{43}\xi + \bar B_{44}x \end{aligned}\right\} \tag{1}$$

where $\bar\eta \in \mathbf{R}^{m-1}$, $\sigma, \xi \in \mathbf{R}$, μ is the small parameter, the function $U(x, u)$ is the smooth function of its arguments satisfying the inequality

$$U_1|\xi| \le \xi U(x, u(\xi)) \le U_2|\xi|, \qquad U(\xi) = \operatorname{sign}\xi \qquad (U_2 > U_1 > 0)$$

for all $|\xi| < R, x \in \mathbf{R}^n$, $\bar B_{ij}$ are matrices smoothly depending on μ.

The specific features of the system (1) are:

- solutions of the system (1) (see Filippov 1988) are determined uniquely;
- only the second order sliding mode can occur, and the dynamics in this mode is determined by the equations

$$\left.\begin{aligned} \mu\, \mathrm{d}\bar\eta/\mathrm{d}t &= \bar B_{11}\bar\eta + \bar B_{14}x \\ \mathrm{d}x/\mathrm{d}t &= \bar B_{41}\bar\eta + \bar B_{44}x \end{aligned}\right\} \tag{2}$$

When $\bar B_{11}(0)$ is the Hurwitz matrix, then the system (2), has the asymptotically stable slow-motions integral manifold

$$\bar\eta = h(x, \mu) = \mathcal{H}(\mu)x = (-\bar B_{11}^{-1}(0)\bar B_{14}(0) + O(\mu))x \tag{3}$$

System (2) dynamics on this manifold is described by equations

$$\mathrm{d}x/\mathrm{d}t = (\bar B_{41}\mathcal{H}(\mu) + \bar B_{44})x \tag{4}$$

In this section the sufficient conditions are found ensuring that the manifold (3) is the slow-motions integral manifold of the original SP2SM (1) and reductions principle theorem ensuring the equivalence for stability investigating problems for systems (1) and (4) is proved.

2.2. *SP2SM in the small vicinity of the slow-motions manifold*

After the substitution of the variables $\eta = \bar\eta - h(x, \mu)$ the system (1) takes the form

$$\left.\begin{aligned} \mu\, \mathrm{d}\eta/\mathrm{d}t &= B_{11}\eta + B_{12}\sigma + B_{13}\xi \\ \mu\, \mathrm{d}\sigma/\mathrm{d}t &= B_{21}\eta + B_{22}\sigma + B_{23}\xi + B_{24}x + bU(x, u(\xi)) \\ \mu\, \mathrm{d}\xi/\mathrm{d}t &= \sigma \\ \mathrm{d}x/\mathrm{d}t &= B_{41}\eta + B_{42}\sigma + B_{43}\xi + B_{44}x \end{aligned}\right\} \tag{5}$$

where matrices B_{ij} depend on μ. It is necessary to remark that $B_{1j}(0) = \bar B_{1j}(0), j = 1, 2, 3$, $B_{22}(0) = \bar B_{22}(0)$.

2.3. *Exponential stability of fast motions*

Considering the system (5) let us denote as $y = (\eta^{\mathrm{T}}, \sigma, s)^{\mathrm{T}}$, $|y|_* = \sqrt{|\eta|^2 + |\sigma|^2 + |\xi|}$, $(y(t, \mu), \ x(t, \mu))$ system (5) solution with initial conditions

$$y(t_0, \mu) = y_0, \qquad x(t_0, \mu) = x_0$$

Lemma 1: *Suppose that:*

(1°) $\bar B_{11}(0)$ *is Hurwitz matrix,*

(2°) $\bar B_{22}(0) < 0$,

(3°) $b(0) < 0$.

Then there exist constants $K_1 > 0$, $\gamma > 0$ and W, which is some neighbourhood of the origin in the state space of variables $y = (\eta^{\mathrm{T}}, \sigma, \xi)^{\mathrm{T}}$, such that for all $(t_0, y_0, x_0) \in \Omega' = R^+ \times W \times R^n$ and $\mu \in (0, \mu_0]$ the following inequality holds

$$|y(t, \mu)|_* \le K\, \mathrm{e}^{-\gamma(t - t_0)/\mu} \tag{6}$$

2.4. *Decomposition theorem*

Consider only a solution of (5) starting in Ω'. Then the $x(t, \mu)$ coordinate of the solution (5) will be a solution of the initial problem

$$\mathrm{d}x/\mathrm{d}t = \Phi(y(t, \mu), x, \mu), \qquad x = x_0$$

$$\Phi(y(t, \mu), x, \mu) = B_{41}\eta + B_{42}(\mu)\sigma + B_{43}(\mu)\xi + B_{44}(\mu)x$$

Let us represent $x(t, \mu)$ as $x(t, \mu) = \bar x(t, \mu) + \pi x(t, \mu)$ such that $\bar x(t, \mu), \pi x(t, \mu)$ are solutions of equations

$$\mathrm{d}\bar x/\mathrm{d}t = \Phi(t, 0, \bar x, \mu), \qquad \bar x(t_0) = \bar x_0 \tag{7}$$

$$\mathrm{d}\pi x/\mathrm{d}t = \Phi(t, y(t, \mu), \bar x + \pi x, \mu) - \Phi(t, 0, x, \mu) \tag{8}$$

$$\pi x(0) = \pi_0 x, \qquad \bar x_0 + \pi_0 x = x_0 \tag{9}$$

To define the solutions of the problems (7) and (9) it is necessary to choose $(x_0, \pi_0 x)$. The following theorem shows that $(x_0, \pi_0 x)$ can be chosen in such a way that function $\pi x(t, \mu)$ exponentially decreases. For $M = \sup_{\mu\in[0,\mu_0]}\{||B_{4i}||, i = 1, 2, 3, 4\}$ one has

$$|\Phi(t, y, x) - \Phi(t, \bar{y}, \bar{x})| < M(|y - \bar{y}| + |x - \bar{x}|)$$

Theorem 1: *Suppose that for all* $(t, y, x), (t, \bar{y}, \bar{x}) \in \Omega'$ *conditions* (1°)–(3°) *hold, and*

$$\mu M/\gamma < 1, \qquad KM/(\gamma - \mu M) < C \tag{10}$$

Then for any initial points $(t_0, y_0, x_0) \in \Omega'$ *the solutions of the system* (5) *can be represented as a combination of the slow and fast parts in the form*

$$(y(t, \mu), x(t, \mu)) = (0, \bar{x}(t, \mu)) + (\pi y(t, \mu), \pi x(t, \mu))$$

so $\bar{x}(t, \mu)$ *is the solution of equation* (7) *with initial conditions* $\bar{x}(0) = \bar{x}_0$ *while* $x_0 = \bar{x}_0 + O(\mu)$. *The fast part of this solution* $\{\pi y(t, \mu), \pi x(t, \mu)\}$ *satisfies the inequality*

$$\mu|\pi y(t, \mu)| + |\pi x(t, \mu)| < \mu(C + K)\,\mathrm{e}^{-\gamma(t-t_0)/\mu}$$

2.5. *Reduction principle*

Theorem 1 and exponential increase of

$$\mu|\pi y(t, \mu)| + |\pi x(t, \mu)|$$

yield the following reduction principle theorem.

Theorem 2: *If under the conditions of Theorem 1 the function* $\bar{x}(t)$ *is the solution of the system* (7) *then* $(0, 0, 0, x(t, \mu))$ *is the solution of the system* (5) *and this solution will be stable (unstable, asymptotically stable) if and only if* $\bar{x}(t)$ *is stable (unstable, asymptotically stable).*

3. Influence of inertial sensors dynamics on behaviour of sliding mode control systems

3.1. *Problem statement*

Suppose that the regular form (see, e.g. Utkin 1992) of the sliding mode control system has the form

$$\left.\begin{aligned} \mathrm{d}s/\mathrm{d}t &= A_1 s + A_2 x + bU(x, u(s)) \\ \mathrm{d}x/\mathrm{d}t &= A_3 s + A_4 x \end{aligned}\right\} \tag{11}$$

where $s \in R, x \in R^n$, s is the output of (11), and the relay control function $u(s) = \mathrm{sign}(s)$ which ensures the presence of stable first order sliding mode on the surface $s = 0$ with a finite input time, is designed. The sliding mode dynamics in system (11) is described by system

$$\mathrm{d}\bar{x}/\mathrm{d}t = A_4\bar{x} \tag{12}$$

We suppose that the behaviour of the switching surface $s = 0$ (input) is measured by an inertial sensor. The behaviour of the inertial sensor is described by equation

$$\mu\,\mathrm{d}z/\mathrm{d}t = Nz + Hs + Gx \tag{13}$$

where $z \in R^m$ is the inertial sensor output vector, μ is the small parameter determining the time constant of measurement. This means that the switching surface of relay control in the system (11) and (13) is the surface $S = Cz$, but not the surface $s = 0$. Consequently, the dynamics of the sliding mode system (11) with inertial sensor (13) is described by the system

$$\left.\begin{aligned} \mu\,\mathrm{d}z/\mathrm{d}t &= Nz + Hs + Gx \\ \mathrm{d}s/\mathrm{d}t &= A_1 s + A_2 x + bU(x, u(S)) \\ \mathrm{d}x/\mathrm{d}t &= A_3 s + A_4 x \end{aligned}\right\} \tag{14}$$

Computing the derivatives of switching surface S according to (14) we obtain

$$\begin{aligned} \mu\,\mathrm{d}S/\mathrm{d}t &= C(Nz + Hs + Gx) \\ \mu^2\,\mathrm{d}^2S/\mathrm{d}t^2 &= C(N(Nz + Hs + Gx) \\ &\quad + \mu H(A_1 s + A_2 x + bU(x, u(S))) \\ &\quad + \mu G(A_3 s + A_4 x)) \end{aligned}$$

Consequently under condition $CHb \neq 0$ in the system (14) only the second order sliding modes can occur.

Suppose that the 'static condition'

$$CN^{-1}G = 0$$

is satisfied. This means that, when the sensor is ideal ($\mu = 0$), the system (14) coincides with system (11).

3.2. *Choice of initial conditions*

It is reasonable to consider the systems with stable sensors. Suppose that matrix N is stable and therefore non-singular. This means that a solution of the system (14) for the time $o(\mu)$ reaches the neighbourhood of the manifold $z_0 = -N^{-1}(Hs + Gx)$. Due to this fact we shall examine only the system with the initial conditions

I1. $z(t_0) = -N^{-1}(Hs(t_0) + Gx(t_0)) = O(\mu)$

The relay control function $U(x, u(s))$ ensures existence of a stable first order sliding mode in (11). Therefore, solutions of (11) reach the surface $s = 0$ in a finite time. Consequently, before the first switching moment system (14) is smooth. Then according to the boundary layer method (see, e.g. Vasil'eva *et al.* 1995) the solutions of the system (14) reach the $O(\mu)$ neighbourhood of the surfaces $s = 0$ and $S = 0$ in a finite time. It allows to examine only solutions of the system (14) with initial conditions

I2. $s(t_0) = O(\mu)$

I3. $S(t_0) = O(\mu)$

Below we will show that system (11) solutions satisfying the initial conditions I1– I3 will not leave the small neighbourhoods of the ideal and real switching surfaces and will describe the oscillations of the ideal switching surface.

3.3. *Decomposition algorithm for sliding mode systems with inertial sensors*

To use the results of § 2 for investigation of systems (11) and (13) the following four-step decomposition algorithm is useful:

1. To study oscillations in the direction orthogonal to the switching surface and its derivative it is necessary to single out variable S as one of the system's (11) coordinate. Let us suppose that the last coordinate of the vector C is non-degenerate. Then we can substitute the variable S instead of the last coordinate of the vector z in the system (13). Then the system (13) takes the form

$$\left.\begin{aligned} \mu\, d\bar{z}/dt &= N_{11}\bar{z} + N_{12}S + H_1 s + G_1 x \\ \mu\, dS/dt &= N_{21}\bar{z} + N_{22}S + H_2 s + G_2 x \end{aligned}\right\} \quad (15)$$

where the vector $\bar{z}$ consists of the $(m-1)$ first coordinates of the vector z.

2. Let us eliminate the variable s from the first equation of (15). For this purpose we make the variable replacement

$$\hat{z} = \bar{z} - H_1 H_2^{-1} S$$

in the system (15). Then the system (15) takes the form

$$\left.\begin{aligned} \mu\, d\hat{z}/dt &= D_{11}\hat{z} + D_{12}S + G_3 x \\ \mu\, dS/dt &= D_{21}\hat{z} + D_{22}S + H_2 s + G_4 x \end{aligned}\right\} \quad (16)$$

3. Taking into account conditions I1–I3, consider only system (16) solutions for which the right-hand sides of equation (16), and the variables s and S are small. It allows the introduction of variables $\bar{\eta}$, σ and ξ instead of $\hat{z}$, S and s in system (16) according to formulae

$$\mu\bar{\eta} = \hat{z} + D_{11}^{-1}(D_{12}S + G_3 x), \qquad \mu\xi = S$$

$$\mu\sigma = \mu\, dS/dt = D_{21}\hat{z} + D_{22}S + H_2 s + G_4 x$$

Then system (14) takes the form

$$\left.\begin{aligned} \mu\, d\bar{\eta}/dt &= M_{11}\bar{\eta} + M_{12}\sigma + M_{13}\xi + M_{14}x \\ \mu\, d\sigma/dt &= M_{21}\bar{\eta} + M_{22}\sigma + M_{23}\xi + M_{24}x \\ &\quad + dU(x, u(\xi)) \\ \mu\, d\xi/dt &= \sigma \end{aligned}\right\} \quad (17)$$

$$dx/dt = M_{21}\bar{\eta} + M_{22}\sigma + M_{23}\xi + M_{24}x \quad (18)$$

Only the second order sliding modes can occur in system (17) and (18) for $d \neq 0$. Dynamics in this mode are described by equations

$$\left.\begin{aligned} \mu\, d\bar{\eta}/dt &= M_{11}\bar{\eta} + M_{14}x \\ dx/dt &= M_{41}\bar{\eta} + M_{44}x \end{aligned}\right\} \quad (19)$$

4. Let us find the slow motions integral manifold for system (19) in the form

$$\bar{\eta} = \mathcal{H}(\mu)x = -M_{11}^{-1}(M_{14} + \mu M_{11}^{-1}M_{14} \times (M_{44} - M_{41}M_{11}^{-1}M_{14}) + \cdots)x$$

where $\mathcal{H}(\mu)$ is derived from the system

$$M_{11}\mathcal{H}(\mu) + M_{14} = \mu(M_{41}\mathcal{H}(\mu) + M_{44})$$

Motions in this manifold are described by the equations

$$d\bar{x}/dt = (M_{11}^{-1}\mathcal{H}(\mu) + M_{14})\bar{x} \quad (20)$$

For $\mu = 0$ system (20) coincides with system (12) describing the sliding motions for the reduced system. For $\mu > 0$ system (20) allows to take into account the presence of the fast sensors in sliding mode system, and describes the slow dynamics in (17) and (18) up to the fast decreasing exponent. Moreover, the following theorem is true.

Theorem 3: *Assume:*

(i) M_{11}*-stable matrix.*

(ii) $M_{22} < 0, \quad d < 0.$

Then the slow-motions integral manifold of the system (19) *describing the behaviour of the system in the domain of second order sliding is a stable integral manifold of slow motions for system* (17) *and* (18). *The coordinates* $\bar{\eta}(t,\mu)$, $\sigma(t,\mu)$, $\xi(t,\mu)$ *and* $x(t,\mu)$ *of solution* (17) *and* (18) *may be represented as a sum*

$$\begin{aligned} (\bar{\eta}(t,\mu), \sigma(t,\mu), \xi(t,\mu), x(t,\mu)) &= (\mathcal{H}(\mu)\bar{x}(t,\mu), 0, 0, \bar{x}(t,\mu)) \\ &\quad + (\pi\eta(t,\mu), \sigma(t,\mu), \xi(t,\mu), \pi x(t,\mu)) \end{aligned}$$

where $x(t,\mu)$ *is the solution of* (20) *with the initial condition*

$$\bar{x}(t_0) = x^0 + O(\mu)$$

In this case, for some $L, \gamma > 0$ *and for all* $t > t_0$ *the following inequality is true*

$$\mu|(\pi\eta(t,\mu), \sigma(t,\mu), \xi(t,\mu))| + |\pi x(t,\mu))| < \mu L\, e^{-\gamma(t-t_0)/\mu}$$

3.4. *Formula for ideal switching surface deviations*

From Theorem 3 it follows that oscillations in the direction orthogonal to the switching surface decrease exponentially fast and oscillations of the ideal switching surface s may be found from the formula

$$s = -\mu H_2^{-1} D_{21} \mathcal{H}(\mu)$$

with an accuracy up to a fast decreasing exponent.

3.5. *Example*

Let us suppose that in the control system

$$\left. \begin{array}{c} \mathrm{d}x_1/\mathrm{d}t = x_3 - u(s), \qquad \mathrm{d}x_2/\mathrm{d}t = x_1 + x_2 \\ \mathrm{d}x_3/\mathrm{d}t = x_2 \end{array} \right\} \qquad (21)$$

the relay control $u(s) = \mathrm{sign}(s)$, $s = x_1 + x_2 + x_3$ is designed. Substituting in (21) the variable s instead of variable x_2 we will have

$$\left. \begin{array}{c} \mathrm{d}s/\mathrm{d}t = s + x_2 - \mathrm{sign}\, s, \qquad \mathrm{d}x_2/\mathrm{d}t = s - x_3 \\ \mathrm{d}x_3/\mathrm{d}t = x_2 \end{array} \right\} \qquad (22)$$

Motions in sliding mode occurring in (22) are described by the system

$$\mathrm{d}x_2/\mathrm{d}t = -x_3, \; \mathrm{d}x_3/\mathrm{d}t = x_2 \qquad (23)$$

The zero solution of the system (23) is stable but not asymptotically. Now let us suppose that the behaviour of variables x_1, x_2 and x_3 is measured with the help of a fast inertial sensor whose behaviour is determined by the equations

$$\left. \begin{array}{c} \mu\, \mathrm{d}z_1/\mathrm{d}t = -z_1 + x_1, \qquad \mu\, \mathrm{d}z_2/\mathrm{d}t = -z_2 + x_2 \\ \mu\, \mathrm{d}z_3/\mathrm{d}t = -a(z_3 - x_3) \end{array} \right\} \qquad (24)$$

Then the real switching surface is

$$S = z_1 + z_2 + z_3$$

Let us introduce a new variable $\bar{\eta} = (z_3 - x_3)/\mu$. In this case we have the following equations for S and $\bar{\eta}$

$$\mu\, \mathrm{d}S/\mathrm{d}t = -S + s + (1 - a)(z_3 - x_3)$$

$$\mu\, \mathrm{d}\bar{\eta}/\mathrm{d}t = -a\bar{\eta} - x_2$$

This means that we may describe the oscillations of 'ideal' switching surface with precision up to the fast decreasing exponent with the help of the formula

$$s = \mu(1 - a)x_2/a \qquad (25)$$

The slow motions in (21) and (24) are described by the equations

$$\left. \begin{array}{c} \mathrm{d}x_2/\mathrm{d}t = -x_3 + \mu(1 - a)x_2/a + \cdots \\ \mathrm{d}x_3/\mathrm{d}t = x_2 \end{array} \right\} \qquad (26)$$

The zero solution of the systems (26), and (21) and (24) is asymptotically stable for $a > 1$ and unstable for $0 < a < 1$. Following this approach, in a critical case, when the linear part of the sliding mode equations has a critical spectrum, it is necessary to correct the equations of sliding, taking into account the presence of inertial sensors in the system. The presence of such sensors may change the behaviour of the system from stability to asymptotic stability or instability. The results of simulations for solutions of system (21) behaviour with initial conditions $z_1(0) = -1$, $z_2(0) = 0$, $z_3(0) = 1$, $x_1(0) = -1$, $x_2(0) = 0$, $x_3(0) = 1$ and $\mu = 0.2$ are shown in figures 1–4.

4. Conclusions

The problem of chattering in sliding mode control systems with inertial sensors is analysed. The behaviour of such systems is described by singularly perturbed systems with higher order sliding modes. For the sliding mode control systems with inertial sensors whose behaviour is described by SP2SM:

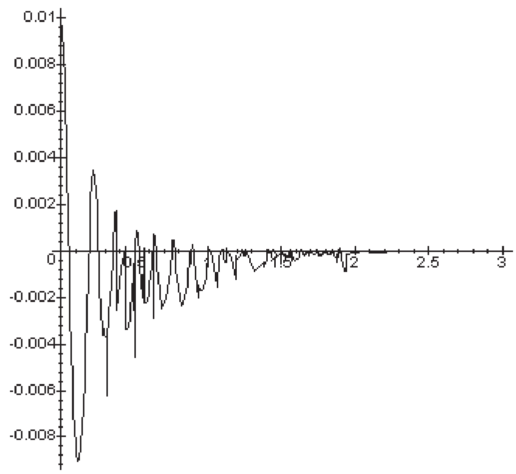

Figure 1. Exponential decreasing of the real switching surface S.

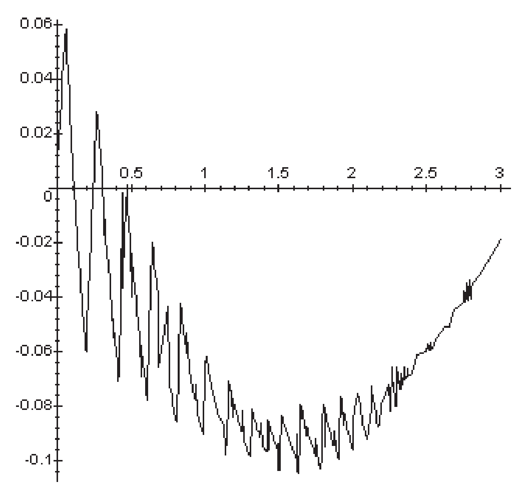

Figure 2. Deviation of the ideal switching surface s.

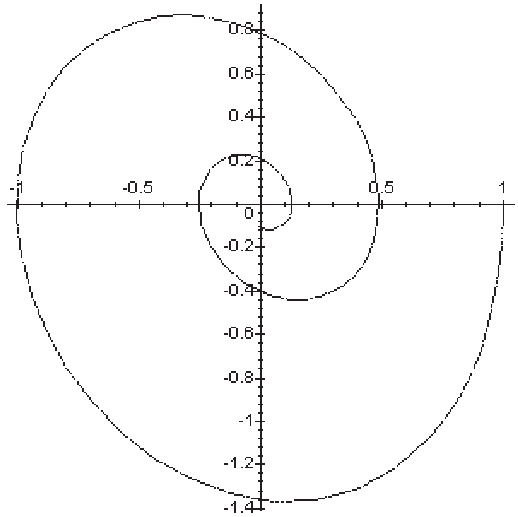

Figure 3. Asymptotical stability of x_2, x_3 for $a = 2$.

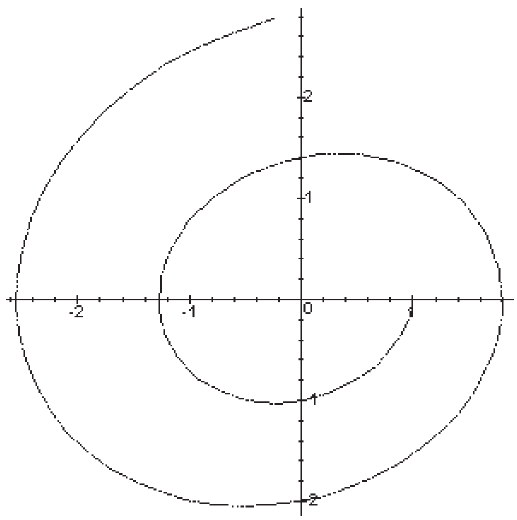

Figure 4. Instability of x_2, x_3 for $a = 0.5$.

a. The concept of slow-motions integral manifolds is revised.
b. The sufficient conditions are found ensuring the following structure of dynamics:
   - the oscillations developed in the direction orthogonal to the switching surface designed according to the measuring results. The amplitude of these oscillations exponentially decreases, time intervals between switches vanish and the frequency of such oscillations is infinitely increasing;
   - the oscillations in the second order sliding mode, which are described by smooth singularly perturbed system of differential equations: the slow-motions integral manifold of this system is the stable slow-motions integral manifold of the original system.
c. The formula for asymptotic analysis of oscillations of the ideal switching surface is derived.
d. The algorithm for correction of the sliding mode equation is proposed. In the case when the linear part of sliding mode equations has a critical spectrum, it is obligatory to correct the equations of sliding motions in order to take into account the presence of fast inertial sensors in the system, because the presence of such devices may cause a change of the system behaviour from stability to asymptotic stability or instability.

It can be concluded that:

(1) By designing the sliding mode control systems with fast inertial sensors it is possible practically to avoid chattering (the amplitude of chattering is decreasing exponentially fast). For this purpose it is necessary to ensure that the complete model of the sliding mode system taking into account the actuators and sensors has the relative degree two.
(2) Unfortunately in the case when the complete model of system is of relative degree more than two chattering occurs. To analyse this phenomenon a corresponding revision of the averaging technique (Fridman 2001) is needed.

Appendix

A.1. *Proof of decomposition theorem*

To prove Lemma 1 consider the Lyapunov function

$$E = \eta^{\mathrm{T}} S\eta + \sigma^2 - \xi[2bU(x, u(\xi)) + B_{22}\sigma + 2B_{23}\xi$$
$$+ 2\eta^{\mathrm{T}} SB_{12} + 2B_{21}\eta]$$

where the matrix $S(\mu), \mu \in [0, \mu_0]$ is the positive definite solution to the matrix equation $S^{\mathrm{T}} B_{11} + B_{11} S = Q$, Q is the positive definite matrix. Then there are some constants $\kappa_2 > \kappa_1 > 0$ such that the inequality

$$\kappa_1 |\eta|^2 \leq \eta^* S\eta \leq \kappa_2 |\eta|^2 \quad (27)$$

is true uniformly for $\mu \in [0, \mu_0]$. Taking into account conditions (1°)–(3°) and inequality (27) one can conclude the following estimation for the Lyapunov function

$$\kappa_3 |y|_*^2 \leq E(t, y, x, \mu) \leq \kappa_4 |y|_*^2 \qquad (\kappa_4 > \kappa_3 > 0) \quad (28)$$

which is true for $(t, y, x, \mu) \in \mathbf{R}^+ \times \mathbf{R}^n \times \mathcal{U}_1 \times (0, \mu_0)$, where $\mathcal{U}_\infty$ is some neighbourhood of zero in the state space of variables $y = (\eta, \sigma, \xi)$. In this case

$$\mathrm{d}E/\mathrm{d}\tau = -|\eta|^2 + B_{22}\sigma^2 - \xi B_{22}[bU(x, u(\xi)) +$$
$$+ B_{22}\sigma + 2\eta^* SB_{12} + 2B_{21}\eta]$$

This means that

$$\kappa_5|y|_*^2 \le -\mathrm{d}E/\mathrm{d}\tau \le \kappa_6|y|_*^2 \qquad (\kappa_6 > \kappa_5 > 0) \tag{29}$$

which is true at some neighbourhood $\mathcal{U}_\in$ of the origin in the state space of y.

From (28) and (29) one can conclude that there are some constants $\kappa_8 > \kappa_7 > 0$, in the neighbourhood of the origin $W = \mathcal{U}_\infty \cap \mathcal{U}_\in$ such that for any $\mu \in [0, \mu_0]$ the following inequality

$$\kappa_7 E \le -\mathrm{d}E/\mathrm{d}\tau \le \kappa_8 E \tag{30}$$

is true. Lemma 1 follows from inequality (30).

A.2. *Proof of decomposition theorem*

Consider the system (7) and (8). Let us design an integral manifold of the system (7) and (8) in the form $\mathcal{S} = \{(t, x, \pi x) \in \mathbf{R}^+ \times \mathbf{R}^n \times \mathbf{R}^n\colon \pi x = \mathcal{F}(t, x, \mu)\}$, where the function $\mathcal{F}(t, x, \mu)$ is continuous on $\mathbf{R}^+ \times \mathbf{R}^n \times [0, \mu_0]$ and the following inequality is true

$$\sup|\exp(\gamma t/\mu)\mathcal{F}(t, x, \mu)| < \mu d, \qquad (t, x) \in \mathbf{R} \times \mathbf{R}^n \tag{31}$$

The constant $d > 0$ in (31) will be defined later. Denote as $\mathcal{U}$ the metric space of continuous functions $\mathbf{R}^+ \times \mathbf{R}^n \times [0, \mu_0] \to \mathbf{R}^n$, satisfying the inequality (31) with the metric $\rho(\mathcal{F}, \bar{\mathcal{F}}) = \sup|\exp(\gamma t/\mu)(\mathcal{F}(t, x, \mu) - \bar{\mathcal{F}}(t, x, \mu))|$, for $(t, x, \mu) \in \mathbf{R}^+ \times \mathbf{R}^n \times [0, \mu_0]$. The space $\mathcal{U}$ is a complete metric space. The function $\pi x = \mathcal{F}(t, x, \mu) \in \mathcal{U}$ is the solution of the equation

$$\mathcal{F} = \mathcal{P}(\mathcal{F}) \tag{32}$$

$$\begin{aligned} \mathcal{P}(\mathcal{F})(t, \hat{x}, \mu) = -\int_t^\infty &[\Phi(y(\theta, \mu), \phi(\theta, \mu) \\ &+ \mathcal{F}(\theta, \phi(\theta, \mu), \mu), \mu) \\ &- \Phi(0, \phi(\theta, \mu), \mu)]\, \mathrm{d}\theta, \end{aligned}$$

where $\phi(\theta, \mu)$ is the solution of Cauchy problem $\mathrm{d}\phi/\mathrm{d}\theta = \Phi(0, \phi, \mu)$, $\phi(t) = \hat{x}$. Let us show that operator $\mathcal{P}$ from (32) transforms $\mathcal{U}$ into itself. Taking into account (31) and (32), one can conclude that

$$\begin{aligned} &|\exp(\gamma t/\mu)\mathcal{P}(\mathcal{F})(t, \hat{x}, \mu)| \\ &\quad \le M \exp(\gamma t/\mu) \int_t^\infty [|\mathcal{F}(\theta, \phi(\theta, \mu), \mu)| + |y(\theta, \mu)|]\, \mathrm{d}\theta \\ &\quad < \frac{M}{\gamma}[\mu d + C|y_0|_*] \end{aligned}$$

Now it is possible to choose such d that for any $y_0 \in W$ the inequality $(M/\gamma)[\mu d + C|y_0|_*] \le d$ is true. This means that operator $\mathcal{P}$ transforms the space $\mathcal{U}$ into itself. Similarly,

$$\begin{aligned} &\exp(\gamma t/\mu)|\mathcal{P}(\mathcal{F})(t, \hat{x}, \mu) - \mathcal{P}(\bar{\mathcal{F}})(t, \hat{x}, \mu)| \\ &\quad \le \exp(\gamma t/\mu) \int_t^\infty |\Phi(\phi(\theta, \mu) + \mathcal{F}, y(\theta, \mu), \mu) \\ &\qquad - \Phi(\phi(\theta, \mu) + \bar{\mathcal{F}}, y(\theta, \mu), \mu)|\, \mathrm{d}\theta \\ &\quad \le \exp(\gamma t/\mu) \int_t^\infty M|\mathcal{F} - \bar{\mathcal{F}}|\, \mathrm{d}\theta \le \mu \frac{M}{\gamma} \rho(\mathcal{F}, \bar{\mathcal{F}}) \end{aligned}$$

which means that the operator $\mathcal{P}$ is a contraction operator on $\mathcal{U}$. Then, the operator $\mathcal{P}$ has the unique fixed point corresponding to the function $\pi x = \mathcal{F}(t, \hat{x}, \mu)$. Moreover, from (31) one can conclude that the inequality

$$|\mathcal{F}(t, \hat{x}, \mu))| < \mu\, d \exp(-\gamma t/\mu)$$

holds for all $(t, \hat{x}, \mu) \in \mathbf{R}^+ \times \mathbf{R}^n \times [0, \mu_0]$.

References

Anosov, D. V., 1959, On stability of equilibrium points of relay systems. *Automatica i telemechanica (Automation and Remote Control)*, **10**, 135–149 (Russian).

Bartolini, G., Ferrara, A., and Usai, E., 1998, Chattering avoidance by second order sliding mode control. *IEEE Transactions on Automatic Control*, **43**, 241–246.

Bondarev, A. G., Bondarev, S. A., Kostylyeva, N. Ye., and Utkin, V. I., 1985, Sliding modes in systems with asymptotic state observers. *Automation and Remote Control*, **46**, 679–684.

Filippov, A. F., 1988, *Differential Equations with Discontinuous Right Hand Side* (Dordrecht: Kluwer Publishers).

Fridman, L. M., 1990, Singular extension of the definition of discontinuous systems and stability. *Differential Equations*, **26**, 1307–1312.

Fridman, L., 2001, An averaging approach to chattering. *IEEE Transactions of Automatic Control*, **46**, 1260–1265.

Fridman, L., and Levant, A., 1996, Higher order sliding modes as the natural phenomenon in control theory. In F. Garafalo and L. Glielmo (Eds) *Robust Control via Variable Structure and Lyapunov Techniques, Variable Structure Systems, Sliding Mode and Nonlinear Control*, Lecture Notes in Control and Information Sciences, **217** (London: Springer Verlag), pp. 107–133.

Shkolnikov, I. A., and Shtessel, Y. B., 2002, Tracking a class of nonminimum phase systems with nonlinear internal dynamics via sliding mode control using method of system center. *Automatica*, **38**, 837–842.

Sira-Ramires, H., 1989, Sliding regimes in general non-linear systems: a relative degree approach. *International Journal of Control*, **50**, 1487–1506.

Utkin, V. I., 1992, *Sliding Modes in Control and Optimization* (London: Springer Verlag).

Vasil'eva, A. B., Butuzov, V. F., and Kalachev, L. V., 1995, *The Boundary Function Method for Singular Perturbation Problems* (Philadelphia: SIAM).