Homogeneity based Uniform Stability Analysis for Time-Varying Systems

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Abstract—**The uniform stability notion for a class of nonlinear time-varying systems is studied using the homogeneity framework. It is assumed that the system is weighted homogeneous considering the time variable as a constant parameter, then several conditions of uniform stability for such a class of systems are formulated. The results are applied to the problem of adaptive estimation for a linear system.**

Index Terms—**Time-varying systems, Stability of nonlinear systems, Adaptive control**

I. INTRODUCTION

THE global behavior of trajectories for homogeneous time-invariant dynamical systems can be evaluated based on their behavior on a suitably defined sphere around the origin [1]. Thus, the local and global behaviors of homogeneous systems are the same. This property has been found useful for stability analysis (see, *e.g.* [1], [2], [3], [4] and [5]), approximation of system dynamics (see, *e.g.* [6] and [7]), high-order sliding-modes [8], stabilization (see, *e.g.* [9], [10], [11], [12] and [13]), and estimation (see, *e.g.* [2] and [7]). It has been shown that for stability/instability analysis, Lyapunov function of a homogeneous system can be chosen homogeneous (see, *e.g.* [5], [14] and [15]). Thus, the numerical analysis and design of homogeneous systems may be simpler since, for example, a Lyapunov function has to be only constructed on a sphere (on the whole state space it can extended using homogeneity). In addition, the homogeneous systems have certain intrinsic robustness properties (see, *e.g.* [16] and [17]).

In many cases the system dynamics is perturbed by exogenous disturbances, whose known parts can be modeled by some time functions, then another class of models arise: time-varying dynamical systems. Parameters of these disturbances (the rate of convergence or the main frequency) influence a lot on the system stability. For example, a nonlinear system can be stable for one exponentially converging disturbance and unstable with respect to another one, another example is the resonance phenomenon in linear systems. Moreover, in the adaptive control theory (see, *e.g.* [18] and [19]), the design of the adaptive law is crucial for the stability properties of the adaptive controller. Nevertheless, the adaptive law introduces a multiplicative nonlinearity that makes the closed-loop plant nonlinear and time-varying. Because of this, the analysis and understanding of the stability and robustness of adaptive control schemes (nonlinear time-varying) are more challenging.

Due to robustness properties of homogeneous systems it would be interesting to apply this concept for time-varying systems. An extension of the homogeneity concept to time-varying systems has been given in [20] and [21], where in the latter a re-parametrization of time has also been required together with the state dilation. In this work, the weighted homogeneity theory is applied for the system dynamics considering the time variable as a constant parameter (it slightly differs from [21] and is similar to [20]).

Establishing stability properties, it is also important to quantify the rate of convergence in the system: exponential, asymptotic, finite-time or fixed-time (see, *e.g.* [12], [22], [23], [24] and [25]). Frequently, the homogeneity theory is used to establish finite-time or fixed-time stability (see, *e.g.* [2], [9], [25] and [26]): for example, if a system is globally asymptotically stable and homogeneous of negative degree, then it is finite-time stable. In this work we will also address the question of finite-time stability existence for time-varying systems.

The outline of this work is as follows. The preliminary definitions and the homogeneity framework are given in Section 2. The property of scaling of solutions for some class of homogeneous time-varying systems is presented in Section 3. The conditions of finite-time stability are given in Section 4. Application of the developed theory to the problem of stability and convergence analysis for an adaptive estimator is considered in Section 5.

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II. PRELIMINARIES

Consider a time-varying differential equation [27]:

$$\frac{dx(t)}{dt} = f(t, x(t)), \ t \ge t_0, \ t_0 \in \mathbb{R}, \qquad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector; $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function with respect to x and measurable with

respect to t, and $f(t, 0) = 0$. It is assumed that solution of the system (1) for an initial condition $x_0 \in \mathbb{R}^n$ at time instant $t_0 \in \mathbb{R}$ is denoted as $x(t, t_0, x_0)$ and it is defined on some finite time interval $[t_0, t_0 + T)$ where $0 \leq T < \infty$ (the notation $x(t)$ will be used to reference $x(t, t_0, x_0)$ if the origin of x_0 and t_0 is clear from the context).

A continuous function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{K}$ if it is strictly increasing and $\sigma(0) = 0$; it belongs to class $\mathcal{K}_\infty$ if it is also unbounded. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{KL}$ if, for each fixed s, $\beta(r, s) \in \mathcal{K}$ with respect to r and, for each fixed r, $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$. Denote a sequence of integers $1, ..., m$ as $\overline{1, m}$.

A. Stability definitions

Let Ω be an open subset of $\mathbb{R}^n$, such that $0 \in \Omega$.

Definition 1. [27] *At the steady state $x = 0$ the system* (1) *is said to be*[1]

- *US if for any $t_0 \in \mathbb{R}$ and $\epsilon > 0$ there is $\delta(\epsilon)$ such that for any $x_0 \in \Omega$, if $|x_0| \leq \delta(\epsilon)$ then $|x(t, t_0, x_0)| \leq \epsilon$ for all $t \geq t_0$;*
- *UAS if it is US and for any $t_0 \in \mathbb{R}$, $\kappa > 0$ and $\epsilon > 0$ there exists $T(\kappa, \epsilon) \geq 0$ such that for any $x_0 \in \Omega$, if $|x_0| \leq \kappa$ then $|x(t, t_0, x_0)| \leq \epsilon$ for all $t \geq t_0 + T(\kappa, \epsilon)$;*
- *UFTS if it is US and for any $x_0 \in \Omega$ there exists $0 \leq T^{x_0} < +\infty$ such that $x(t, t_0, x_0) = 0$ for all $t \geq t_0 + T^{x_0}$.*[2]

If $\Omega = \mathbb{R}^n$, then $x = 0$ is said to be globally *US (GUS), UAS (GUAS), or UFTS (GUFTS)*, respectively. For US and UAS properties, there exist corresponding formulations in terms of functions from classes $\mathcal{K}_\infty$ and $\mathcal{KL}$, respectively (see, *e.g.* [27]).

B. Homogeneity

For any $r_i > 0$, $i = \overline{1, n}$ and $\lambda > 0$, define the dilation matrix $\Lambda_r(\lambda) = \text{diag}\{\lambda^{r_i}\}_{i=1}^n$ and the vector of weights $r = [r_1, ..., r_n]^T$.

For any $r_i > 0$, $i = \overline{1, n}$ and $x \in \mathbb{R}^n$ the homogeneous norm can be defined as follows:

$$|x|_r = \left(\sum_{i=1}^n |x_i|^{\rho/r_i} \right)^{1/\rho}, \quad \rho = \prod_{i=1}^n r_i.$$

For all $x \in \mathbb{R}^n$, its Euclidean norm $|x|$ is related with the homogeneous one:

$$\underline{\sigma}_r(|x|_r) \leq |x| \leq \bar{\sigma}_r(|x|_r),$$

for some $\underline{\sigma}_r, \bar{\sigma}_r \in \mathcal{K}_\infty$. In the following, due to this "equivalence", stability analysis with respect to the norm $|x|$ will be substituted with analysis for the norm $|x|_r$. The homogeneous norm has an important property that is $|\Lambda_r(\lambda)x|_r = \lambda|x|_r$ for all $x \in \mathbb{R}^n$. Define $\mathbb{S}_r = \{x \in \mathbb{R}^n : |x|_r = 1\}$.

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[1]The acronyms *US, UAS* and *UFTS* correspond to *Uniformly Stable, Uniformly Asymptotically Stable* and *Uniformly Finite-Time Stable,* respectively.

[2]The function $T_0(x_0) = \inf\{T^{x_0} \geq 0 : x(t, t_0, x_0) = 0 \, \forall t \geq t_0 + T^{x_0}\}$ is called the uniform settling time of the system (1).

Definition 2. [1] *The function $g : \mathbb{R}^n \to \mathbb{R}$ is called r-homogeneous ($r_i > 0$, $i = \overline{1, n}$), if for any $x \in \mathbb{R}^n$ the relation*

$$g(\Lambda_r(\lambda)x) = \lambda^d g(x),$$

holds for some $d \in \mathbb{R}$ and all $\lambda > 0$.

The function $f : \mathbb{R}^n \to \mathbb{R}^n$ is called r-homogeneous ($r_i > 0$, $i = \overline{1, n}$), if for any $x \in \mathbb{R}^n$ the relation

$$f(\Lambda_r(\lambda)x) = \lambda^d \Lambda_r(\lambda) f(x),$$

holds for some $d \geq -\min_{1 \leq i \leq n} r_i$ and all $\lambda > 0$. In both cases, the constant d is called the degree of homogeneity.

A dynamical system

$$\dot{x}(t) = f(x(t)), \ t \geq 0, \tag{2}$$

is called r-homogeneous of degree d if this property is satisfied for the vector function f, in the sense of Definition 2. An advantage of homogeneous systems described by nonlinear ordinary differential equations is that any of its solution can be obtained from another solution under the dilation rescaling and a suitable time re-parametrization:

Proposition 1. *[1] Let $x : \mathbb{R}_+ \to \mathbb{R}^n$ be a solution of the r-homogeneous system* (2) *with the degree d for an initial condition $x_0 \in \mathbb{R}^n$. For any $\lambda > 0$ define $y(t) = \Lambda_r(\lambda)x(\lambda^d t)$ for all $t \geq 0$, then $y(t)$ is also a solution of* (2) *with the initial condition $y_0 = \Lambda_r(\lambda)x_0$.*

In order to apply the weighted homogeneity property, introduced for time-invariant systems in Definition 2, to the time-varying systems (1), an extended concept is needed.

Definition 3. [20] *The function $g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ is called r-homogeneous ($r_i > 0$, $i = \overline{1, n}$), if for any $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$ the relation*

$$g(t, \Lambda_r(\lambda)x) = \lambda^d g(t, x),$$

holds for some $d \in \mathbb{R}$ and all $\lambda > 0$.

The function $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is called r-homogeneous ($r_i > 0$, $i = \overline{1, n}$), if for any $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$ the relation

$$f(t, \Lambda_r(\lambda)x) = \lambda^d \Lambda_r(\lambda) f(t, x),$$

holds for some $d \geq -\min_{1 \leq i \leq n} r_i$ and all $\lambda > 0$.

Thus in the time-varying case (1) the homogeneity will be verified considering t as a constant parameter.

III. MAIN RESULTS

This section has three parts. First, an extension of Proposition 1 is presented for time-varying system (1), and some useful tools for uniform stability analysis of nonlinear time-varying systems are introduced. Second, relation with time scaling is analyzed. Third, the links with finite-time stability are investigated.

A. Scaling solutions of homogeneous time-varying systems

Consider the following modification of the system (1), *i.e.*:

$$\frac{dx(t)}{dt} = f(\omega t, x(t)), \ t \geq t_0, \ t_0 \in \mathbb{R}, \qquad (3)$$

for some $\omega > 0$. The parameter ω represents dependence on the convergence rate of time processes in the system or the frequency of time-varying part. For an initial condition $x_0 \in \mathbb{R}^n$ at initial time t_0 denote the corresponding solution of (3) as $x_\omega(t, t_0, x_0)$, thus $x(t, t_0, x_0) = x_1(t, t_0, x_0)$, with $x(t, t_0, x_0)$ a solution of system (1). In this case the following extension of Proposition 1 (a variant of this result has been formulated in the proof of Theorem 2 in [20]) is given.

Proposition 2. *Let $x(t, t_0, x_0)$ be a solution of the r-homogeneous system* (1) *with the degree d for an initial condition $x_0 \in \mathbb{R}^n$ and $t_0 \in \mathbb{R}$. For any $\lambda > 0$ with $\omega = \lambda^d$, the system* (3) *has a solution $y(t, t_0, y_0) = \Lambda_r(\lambda)x(\lambda^d t, \lambda^d t_0, x_0)$, for all $t \geq t_0$ with the initial condition $y_0 = \Lambda_r(\lambda)x_0$.*

Proof. Differentiating $y(t, t_0, y_0)$, it is obtained

$$\begin{aligned}
\frac{dy(t, t_0, y_0)}{dt} &= \frac{d\Lambda_r(\lambda)x(\lambda^d t, \lambda^d t_0, x_0)}{dt}, \\
&= \lambda^d \Lambda_r(\lambda)\frac{dx(\lambda^d t, \lambda^d t_0, x_0)}{d\lambda^d t}, \\
&= \lambda^d \Lambda_r(\lambda)f[\lambda^d t, x(\lambda^d t, \lambda^d t_0, x_0)], \\
&= f[\lambda^d t, \Lambda_r(\lambda)x(\lambda^d t, \lambda^d t_0, x_0)], \\
&= f[\lambda^d t, y(t, t_0, y_0)],
\end{aligned}$$

and $y(t, t_0, y_0)$ is a solution of (3) for $\omega = \lambda^d$. $\square$

For the case $d = 0$, it is recovered that all solutions of (1) are interrelated as in Proposition 1 (as in time-invariant case).

It is well known fact that for the ordinary differential equation (2) local attractiveness implies global asymptotic stability [1]. In the present setting that result has the following correspondence (a similar conclusion also can be found in the proof of Theorem 2 in [20]).

Lemma 1. *Let the system* (1) *be r-homogeneous with degree $d \neq 0$ and GUAS at the origin, i.e. there is a function $\beta \in \mathcal{KL}$ such that*

$$|x(t, t_0, x_0)|_r \leq \beta(|x_0|_r, t - t_0), \quad \forall t \geq t_0,$$

for any $x_0 \in \mathbb{R}^n$ and any $t_0 \in \mathbb{R}$. Then, the system (3) *is GUAS at the origin for any $\omega > 0$ and*

$$|x_\omega(t, t_0, x_0)|_r \leq \beta_\omega(|x_0|_r, t - t_0), \quad \forall t \geq t_0,$$

for any $x_0 \in \mathbb{R}^n$ and any $t_0 \in \mathbb{R}$, where $\beta_\omega(s, t) = \omega^{1/d}\beta(\omega^{-1/d}s, \omega t)$.

Proof. Select $\omega > 0$ and define $\lambda = \omega^{1/d}$, which is well defined for any $\omega > 0$ since $d \neq 0$. Define $y(t, t_0, y_0) = \Lambda_r(\lambda)x(\lambda^d t, \lambda^d t_0, x_0)$ that, according to Proposition 2, is a solution of (3) for the chosen ω and for all $t \geq t_0$ with the

initial condition $y_0 = \Lambda_r(\lambda)x_0$, then

$$\begin{aligned}
|y(t, t_0, x_0)|_r &\leq \lambda |x(\lambda^d t, \lambda^d t_0, x_0)|_r, \\
&\leq \lambda \beta(|x_0|_r, \lambda^d[t - t_0]), \\
&\leq \lambda \beta(\lambda^{-1}|y_0|_r, \lambda^d[t - t_0]), \\
&= \omega^{1/d}\beta(\omega^{-1/d}|y_0|_r, \omega[t - t_0]),
\end{aligned}$$

for all $t \geq t_0$ and for any $y_0 \in \mathbb{R}^n$. $\square$

It is a well known fact that for linear time-varying systems (homogeneous systems of degree $d = 0$) that its stability for some ω does not imply stability for all $\omega \in (0, +\infty)$. For nonlinear homogeneous time-varying systems with degree $d \neq 0$ this is not the case, according to the result of Lemma 1, if they are GUS for some ω, then they preserve the uniform stability for an arbitrary $\omega > 0$. This is an intriguing advantage of "nonlinear" time-varying systems. In addition, it is shown in Lemma 1 that the rate of convergence will be scaled by ω, thus the time of transients in these systems is predefined by the time-varying part, which is not the case for the degree $d = 0$, where the rate of convergence cannot be modified by ω (see Proposition 2)!

Further let us consider several useful consequences of Proposition 2 and Lemma 1.

Corollary 1. *Let the system* (1) *be r-homogeneous with degree $d = 0$ and UAS at the origin into the set $\Omega = B_\rho = \{x \in \mathbb{R}^n : |x|_r \leq \rho\}$, for some $0 < \rho < +\infty$. Then, the system* (1) *is GUAS at the origin.*

Proof. In this case there is a function $\beta \in \mathcal{KL}$ such that

$$|x(t, t_0, x_0)|_r \leq \beta(|x_0|_r, t - t_0), \quad \forall t \geq t_0,$$

for any $x_0 \in B_\rho$ and any $t_0 \in \mathbb{R}$. Take any $\bar{x}_0 \notin B_\rho$, then there is $x_0 \in B_\rho$, with $|x_0|_r = \rho$, such that $\bar{x}_0 = \Lambda_r(\lambda)x_0$, with $\lambda = |\bar{x}_0|_r\rho^{-1}$. By Proposition 2, $x(t, t_0, \bar{x}_0) = \Lambda_r(\lambda)x(t, t_0, x_0)$ is the corresponding solution of (1) and

$$\begin{aligned}
|x(t, t_0, \bar{x}_0)|_r &= \lambda |x(t, t_0, x_0)|_r, \\
&\leq \lambda\beta(|x_0|_r, t - t_0), \\
&= |\bar{x}_0|_r\rho^{-1}\beta_\omega(\rho, t - t_0),
\end{aligned}$$

for all $t \geq t_0$. $\square$

Corollary 2. *Let the system* (3) *be r-homogeneous with degree $d \neq 0$ and UAS at the origin into the set $\Omega = B_\rho = \{x \in \mathbb{R}^n : |x|_r \leq \rho\}$, for a fixed $0 < \rho < +\infty$, for any $\omega > 0$. Then, the system* (3) *is GUAS at the origin, for any $\omega > 0$.*

Proof. In this case, for any $\omega > 0$, there is some function $\beta_\omega \in \mathcal{KL}$ such that

$$|x_\omega(t, t_0, x_0)|_r \leq \beta_\omega(|x_0|_r, t - t_0), \quad \forall t \geq t_0,$$

for any $x_0 \in B_\rho$ and any $t_0 \in \mathbb{R}$, the functions β_ω have a continuous dependence on ω since the solutions of (3) depend continuously on the parameter ω [27]. Note that, for any $\bar{x}_0 \notin B_\rho$ there is $x_0 \in B_\rho$, with $|x_0|_r = \rho$, such that $\bar{x}_0 = \Lambda_r(\lambda)x_0$,

with $\lambda = |\bar{x}_0|_r \rho^{-1}$. Select $\omega = \lambda^{-d}$, then, $x_1(t, t_0, \bar{x}_0) = \Lambda_r(\lambda) x_\omega(\lambda^d t, \lambda^d t_0, x_0)$, by Proposition 2, and

$$|x_1(t, t_0, \bar{x}_0)|_r = \lambda |x_\omega(\lambda^d t, \lambda^d t_0, x_0)|_r,$$
$$\leq \lambda \beta_\omega(|x_0|_r, \lambda^d[t - t_0]),$$
$$= |\bar{x}_0|_r \rho^{-1} \beta_\omega(\rho, |\bar{x}_0|_r^d \rho^{-d}[t - t_0]),$$

for all $t \geq t_0$. Therefore, $|x_1(t, t_0, \bar{x}_0)|_r \leq \rho^{-1} \beta_\omega(\rho, 0)|\bar{x}_0|_r$ for all $t \geq t_0$ and all $\bar{x}_0 \in \mathbb{R}^n$. Take $\delta > \rho$, then $\omega \in [\delta^{-d} \rho^d, 1]$ for $\rho \leq |\bar{x}_0|_r \leq \delta$, and there is $p_{\delta,\rho} = \sup_{\omega \in [\delta^{-d} \rho^d, 1]} \beta_\omega(\rho, 0)$, then $|x_1(t, t_0, \bar{x}_0)|_r \leq \max\{\rho^{-1} p_{\delta,\rho}|\bar{x}_0|_r, \beta_1(|\bar{x}_0|_r, 0)\}$ for all $t \geq t_0$ and all $|\bar{x}_0|_r \leq \delta$, and the system (3), for $\omega = 1$, is GUS. In addition, for any $\kappa > 0$ and $\epsilon > 0$, if $\rho \leq |\bar{x}_0|_r \leq \kappa$, then

$$|x_1(t, t_0, \bar{x}_0)|_r \leq \kappa \rho^{-1} \beta_\omega(\rho, t - t_0),$$

and by the properties of the functions from class $\mathcal{KL}$, there is $\hat{T}(\kappa, \epsilon) \geq 0$ such that $\kappa \rho^{-1} \beta_\omega[\rho, \hat{T}(\kappa, \epsilon)] \leq \epsilon$, for any $\omega \in [\kappa^{-d} \rho^d, 1]$. Consequently, if $|\bar{x}_0|_r \leq \kappa$, then for any $\epsilon > 0$, $|x_1(t, t_0, \bar{x}_0)|_r \leq \epsilon$ for $t \geq t_0 + \max\{\hat{T}(\kappa, \epsilon), \hat{T}(\kappa, \rho) + T_1(\rho, \epsilon)\}$, where $T_1(\kappa, \epsilon)$ is the solution of the equation $\beta_1[\kappa, T_1(\kappa, \epsilon)] = \epsilon$, and therefore, the system (3), for $\omega = 1$, is GUAS. The claim for all $\omega > 0$ follows from Lemma 1. $\square$

Corollary 3. *Let the system* (3) *be r-homogeneous with degree $d > 0$ and UAS at the origin into the set B_ρ for some $0 < \rho < +\infty$, for any $0 \leq \omega \leq \omega_0$, with $0 < \omega_0 < +\infty$. Then, the system* (3) *is GUAS at the origin, for any $\omega > 0$.*

Proof. In this case, again for any $\omega > 0$, there is some function $\beta_\omega \in \mathcal{KL}$ such that

$$|x_\omega(t, t_0, x_0)|_r \leq \beta_\omega(|x_0|_r, t - t_0), \quad \forall t \geq t_0,$$

for any $x_0 \in B_\rho$ and any $t_0 \in \mathbb{R}$, the functions β_ω have a continuous dependence on ω. Take $\bar{x}_0 \notin B_\rho$ and select $1 < \lambda = \rho^{-1}|\bar{x}_0|_r$, then $\bar{x}_0 = \Lambda_r(\lambda) x_0$, for some $x_0 \in B_\rho$, with $\|x_0\|_r = \rho$. Further, according to Proposition 2, $x_{\omega_0}(t, t_0, \bar{x}_0) = \Lambda_r(\lambda) x_\omega(\lambda^d t, \lambda^d t_0, x_0)$ for all $t \geq t_0$, where $\omega = \lambda^{-d} \omega_0 \leq \omega_0$ since $d > 0$ and $\lambda > 1$, and

$$|x_{\omega_0}(t, t_0, x_0)|_r \leq \lambda |x_\omega(\lambda^d t, \lambda^d t_0, x_0)|_r,$$
$$\leq \lambda \beta_\omega(|x_0|_r, \lambda^d[t - t_0]),$$
$$= \rho^{-1} |\bar{x}_0|_r \beta_\omega(\rho, \rho^{-d} |\bar{x}_0|_r^d [t - t_0]).$$

Next, applying similar arguments as in the proof of Corollary 2, the global uniform asymptotic stability at the origin of the system (3) for ω_0 can be substantiated. Using the result of Lemma 1 the claim about global uniform asymptotic stability with respect to all $\omega > 0$ can be recovered. $\square$

Corollary 4. *Let the system* (3) *be r-homogeneous with degree $d < 0$ and UAS with respect to the set B_ρ for some $0 < \rho < +\infty$, for any $0 \leq \omega \leq \omega_0$, with $0 < \omega_0 < +\infty$[3]. Then, the system* (3) *is GUAS at the origin, for any $\omega > 0$.*

Proof. Take $\rho' > \rho$ and $\lambda = \frac{|\bar{x}_0|_r}{\rho'} \leq 1$ for any $\bar{x}_0 \in B_\rho$, then there exists $x_0 \in B_{\rho'}$, $|x_0|_r = \rho'$ such that

$\bar{x}_0 = \Lambda_r(\lambda) x_0$. Define $\omega = \lambda^{-d} \omega_0$, then $\omega \leq \omega_0$, for $\lambda < 1$ and $d < 0$. According to Proposition 2, $x_{\omega_0}(t, t_0, \bar{x}_0) = \Lambda_r(\lambda) x_\omega(\lambda^d t, \lambda^d t_0, x_0)$, for all $t \geq t_0$. Since $|x_\omega(t, t_0, x_0)|_r \leq \sigma_\omega(|x_0|_r)$, for all $t \geq t_0$, then $|x_{\omega_0}(t, t_0, \bar{x}_0)|_r \leq \lambda \sigma_\omega(\lambda^{-1}|\bar{x}_0|_r) = \rho'^{-1}|\bar{x}_0|_r \sigma_\omega(\rho')$, for all $t \geq t_0$ and all $\bar{x}_0 \in B_\rho$. Define $R = \sup_{\omega \in [0, \omega_0]} \sigma_\omega(\rho')$, then $|x_{\omega_0}(t, t_0, \bar{x}_0)|_r \leq \eta(|\bar{x}_0|_r) = \max\{\rho'^{-1}|\bar{x}_0|_r R, \sigma_{\omega_0}(|\bar{x}_0|_r)\}$, for all $t \geq t_0$, for any $\bar{x}_0 \in \mathbb{R}^n$, and the system (3) is GUS for ω_0. Select $\varepsilon > 0$ and take $\lambda = \frac{\eta^{-1}(\varepsilon)}{\rho + \varepsilon} < 1$, then, for any $\bar{x}_0 \in B_\rho \setminus B_{\eta^{-1}(\varepsilon)}$, there exists $x_0 \notin B_\rho$, with $|x_0|_r \leq \lambda^{-1} \rho$, such that $\bar{x}_0 = \Lambda_r(\lambda) x_0$. Since again $x_{\omega_0}(t, t_0, \bar{x}_0) = \Lambda_r(\lambda) x_\omega(\lambda^d t, \lambda^d t_0, x_0)$, for all $t \geq t_0$, and $|x_\omega(\lambda^d t, \lambda^d t_0, x_0)|_r \leq \rho + \varepsilon$, for all $\lambda^d t \geq \lambda^d t_0 + T_{\lambda^{-1}\rho, \varepsilon}^\omega$, then $|x_{\omega_0}(t, t_0, \bar{x}_0)|_r \leq \eta^{-1}(\varepsilon) \leq \varepsilon$, for all $t \geq t_0 + \lambda^{-d} T_{\lambda^{-1}\rho, \varepsilon}^\omega$. For $\bar{x}_0 \in B_{\eta^{-1}(\varepsilon)}$, it is obtained that $|x_{\omega_0}(t, t_0, \bar{x}_0)|_r \leq \varepsilon$, for all $t \geq t_0$, that implies uniform attractivity of the origin for system (3) for ω_0. From these properties the global uniform asymptotic stability for ω_0 is deduced. Finally, the claim of the corollary for any $\omega \in (0, +\infty)$ follows from the conclusion of Lemma 1. $\square$

From this proof it is given that the requirements for $\omega = 0$ can be skipped if the system is stable, and thus, the following corollary arrives.

Corollary 5. *Let the system* (3) *be r-homogeneous with degree $d < 0$ and uniformly asymptotically stable with respect to the set B_ρ for some $0 < \rho < +\infty$, for any $0 < \omega \leq \omega_0$, with $0 < \omega_0 < +\infty$[4]. If the system* (3) *is US, for any $\omega > 0$, then it is GUAS at the origin, for any $\omega > 0$.*

The results presented in this section open a possibility for stability analysis of nonlinear time-varying systems. In particular, they can be interpreted as robustness of the time-invariant homogeneous systems with respect to time-varying perturbations of parameters, *i.e.* if for one value of parameter ω for this time-varying perturbation the system is stable, then it can be stable for any other value of ω.

B. Robustness with respect to time scaling

Consider, for some $a > 0$, the following modification of (1) or (3)

$$\frac{dz(t)}{dt} = af(at, z(t)), \ t \geq t_0, \ t_0 \in \mathbb{R}, \tag{4}$$

where $z \in \mathbb{R}^n$ is the state. Then, $z(t, t_0, x_0) = x(at, at_0, x_0)$, for $t \geq t_0$, is a solution of system (4), for the initial condition $x_0 \in \mathbb{R}^n$ and $x(t, t_0, x_0)$ a solution of system (1). Indeed,

$$\frac{dz(t, t_0, x_0)}{dt} = \frac{dx(at, at_0, x_0)}{dt},$$
$$= a \frac{dx(at, at_0, x_0)}{d\,at},$$
$$= af[at, x(at, at_0, x_0)],$$
$$= af(at, z(t, t_0, x_0)).$$

---

[3]In this case, for each $0 \leq \omega \leq \omega_0$, any $\varepsilon > 0$ and $\kappa \geq 0$, there is $0 \leq T_{\kappa, \varepsilon}^\omega < +\infty$, such that $|x_\omega(t, t_0, x_0)|_r \leq \rho + \varepsilon$, for all $t \geq t_0 + T_{\kappa, \varepsilon}^\omega$, for any $x_0 \in B_\kappa$, and $|x(t, t_0, x_0)|_r \leq \sigma_\omega(|x_0|_r)$, for all $t \geq t_0$, for some function $\sigma_\omega \in \mathcal{K}$, for all $t_0 \in \mathbb{R}$.

[4]For each $0 < \omega \leq \omega_0$, any $\varepsilon > 0$ and $\kappa \geq 0$, there is $0 \leq T_{\kappa, \varepsilon}^\omega < +\infty$, such that $|x_\omega(t, t_0, x_0)|_r \leq \rho + \varepsilon$, for all $t \geq t_0 + T_{\kappa, \varepsilon}^\omega$, for any $x_0 \in B_\kappa$, and $|x(t, t_0, x_0)|_r \leq \sigma_\omega(|x_0|_r)$, for all $t \geq t_0$, for some function $\sigma_\omega \in \mathcal{K}$, for all $t_0 \in \mathbb{R}$.

Thus time scaling (multiplication on a) acts similarly on solutions as the dilation transformation in homogeneous systems, and the following conclusion on the system stability can be obtained.

Lemma 2. *Let the system* (1) *be GUAS at the origin. Then, the system* (4) *is GUAS at the origin, for any $a > 0$, and the rate of convergence in* (4) *is scaled by a, with respect to* (1).

Proof. The conditions of the lemma implies that there is some function $\beta \in \mathcal{KL}$ such that

$$|x(t, t_0, x_0)| \le \beta(|x_0|, t - t_0), \quad \forall t \ge t_0,$$

for any $x_0 \in \mathbb{R}^n$ and any $t_0 \in \mathbb{R}$. By consideration above, for any $x_0 \in \mathbb{R}^n$ and $a > 0$, $z(t, t_0, x_0) = x(at, at_0, x_0)$ is the corresponding solution of (4), then

$$|z(t, t_0, x_0)| = |x(at, at_0, x_0)|,$$
$$\le \beta(|x_0|, a[t - t_0]),$$

for all $t \ge t_0$. $\qquad\square$

Consequently, for a homogeneous system (1), it may be obtained an extension of Proposition 2.

Proposition 3. *Let $x(t, t_0, x_0)$ be a solution of the r-homogeneous system* (1) *with the degree d, for an initial condition $x_0 \in \mathbb{R}^n$, and $t_0 \in \mathbb{R}$. For any $\lambda > 0$ and $a > 0$, the system*

$$\frac{dw(t)}{dt} = a\, f(a\omega t, w(t)),\ t \ge t_0,\ t_0 \in \mathbb{R}, \qquad (5)$$

with $\omega = \lambda^d$, has a solution $w(t, t_0, w_0) = \Lambda_r(\lambda) x(a\lambda^d t, a\lambda^d t_0, x_0)$, for all $t \ge t_0$, with the initial condition $w_0 = \Lambda_r(\lambda) x_0$.

Proof. The proof repeats the arguments of Proposition 2. $\quad\square$

Therefore, for $a = \lambda^{-d}$, the systems (1) and (5) have the same rates of convergence, and their corresponding solutions differ in amplitudes only.

IV. FINITE-TIME STABILITY

Assume that the system (1) is r-homogeneous with degree $d < 0$ and GUAS (according to Lemma 1, the system (3) is also GUAS, for any $\omega > 0$ in this case), assume also that $0 < \mathcal{T} < +\infty$ where

$$\mathcal{T} = \sup_{\omega \in [0,1],\, x_0 \in \mathbb{S}_r} T_{x_0}^\omega,$$
$$T_{x_0}^\omega = \sup_{t_0 \in \mathbb{R}} \arg \inf_{\tau \ge 0} \{|x_\omega(t_0 + \tau, t_0, x_0)|_r \le 0.5|x_0|_r\},$$

in other words, for all $\omega \in [0, 1]$ and all initial conditions $x_0 \in \mathbb{S}_r$ the time of convergence to the ball $B_{0.5}$ is less than $\mathcal{T}$, which is finite and independent in $t_0 \in \mathbb{R}$ (the independence in t_0 follows from uniformity of stability). By Proposition 2, for any $x_0 \in \mathbb{R}^n$ and $\lambda > 0$, the solution of the system (3), for $\omega = \lambda^d$, has the form $x_\omega(t, t_0, x_0) = \Lambda_r(\lambda) x(\lambda^d t, \lambda^d t_0, \Lambda_r^{-1}(\lambda) x_0)$, for all $t \ge t_0$, with the initial conditions x_0. Thus $x(t, t_0, x_0) = \Lambda_r^{-1}(\lambda) x_{\lambda^d}(\lambda^{-d} t, \lambda^{-d} t_0, \Lambda_r(\lambda) x_0)$, for any $x_0 \in \mathbb{R}^n$, any $\lambda > 0$, and all $t \ge t_0$. Next, for any $x_0 \in \mathbb{S}_r$ and the

corresponding trajectory $x(t, t_0, x_0)$ of the system (1), it is possible to define a sequence x_k, $k = 1, 2, \ldots$, such that $x_k = x(t_k, t_0, x_0)$, for some t_k, such that $|x_k|_r = 2^{-k}$ (this sequence is well defined since the origin is attractive for (1)). Let us try to evaluate the upper estimate of the difference between $t_k - t_{k-1}$, for all $k = 1, 2, \ldots$, then obviously $t_1 - t_0 \le \mathcal{T}$, by definition. Next, $x_2 = x(t_2 - t_1, t_1, x_1)$, and for $\lambda = 2$ and some $\tilde{x}_0 \in \mathbb{S}_r$, such that $\tilde{x}_0 = \Lambda_r(2) x_1$, it is obtained

$$x(t_2 - t_1, t_1, x_1) = \Lambda_r^{-1}(\lambda)$$
$$\times x_{\lambda^d}(\lambda^{-d}(t_2 - t_1), \lambda^{-d} t_1, \Lambda_r(\lambda) x_1),$$
$$= \Lambda_r^{-1}(2) x_{2^d}(2^{-d}(t_2 - t_1), 2^{-d} t_1, \tilde{x}_0).$$

Since in (3), the value $\omega = 2^d < 1$, for $d < 0$, if $2^{-d}(t_2 - t_1) \ge \mathcal{T}$, then $|x_{2^d}(2^{-d}(t_2 - t_1), 2^{-d} t_1, \tilde{x}_0)|_r \le 0.5|\tilde{x}_0|_r = 0.5$, and $|x(t_2 - t_1, t_1, x_1)|_r \le 0.25 = |x_2|_r$. Therefore, $t_2 - t_1 \le 2^d \mathcal{T}$, and repeating these arguments, for all $k = 1, 2, \ldots$, it is obtained that $t_k - t_{k-1} \le 2^{(k-1)d} \mathcal{T}$. Finally, the time of convergence of the trajectory $x(t, t_0, x_0)$ to the origin T_{1, x_0} is

$$T_{1, x_0} = \sum_{k=1}^{+\infty} t_k - t_{k-1} \le \sum_{k=1}^{+\infty} 2^{(k-1)d} \mathcal{T},$$
$$= \frac{\mathcal{T}}{2^d} \sum_{k=1}^{+\infty} 2^{kd},$$
$$\le \frac{\mathcal{T}}{2^d} \frac{1}{1 - 2^d},$$

and the system is finite-time convergent from $\mathbb{S}_r$. Since for any initial conditions $x_0 \in \mathbb{R}^n$, the corresponding time of convergence to the origin T_{x_0} is the sum of T_{1, x_0} and $T_{x_0}^1$, where $T_{x_0}^1$ is the time of convergence to $\mathbb{S}_r$ from the given initial conditions x_0, and $T_{x_0}^1$ is a function of initial conditions due to the global uniform asymptotic stability of (1), then the system (1) is globally uniformly finite-time convergent. Since system (1) is stable, it is GUFTS. Thus, the following result has been proven.

Theorem 1. *Let* (1) *be r-homogeneous with degree $d < 0$ and GUAS at the origin, assume also that $0 < \mathcal{T} < +\infty$. Then, the system* (1) *is GUFTS at the origin.*

The obtained result has a rather restrictive condition on the finiteness of $\mathcal{T}$ over all $\omega \in [0, 1]$, but at least it provides a hint on restrictions under which the finite-time stability phenomenon is possible in time-varying systems. In the following, an academic example is introduced in order to show that the rate of convergence can be improved increasing the frequency for the finite-time case.

Consider the following nonlinear time-varying system

$$\dot{x}_1(t) = x_2(t), \qquad (6)$$
$$\dot{x}_2(t) = -k_1 \lceil x_1(t) \rceil^{\frac{\gamma}{2-\gamma}} - k_2 \Gamma(\omega t) \lceil x_2(t) \rceil^\gamma, \qquad (7)$$

where $\lceil \cdot \rceil^\gamma \doteq |\cdot|^\gamma \operatorname{sign}(\cdot)$, with $\gamma \in (0, 1)$, $\Gamma(\omega t) = (1 + \sin^2 \omega t)$, and k_1, $k_2 > 0$ positive constants. The system (6)-(7) can be transformed into the following differential
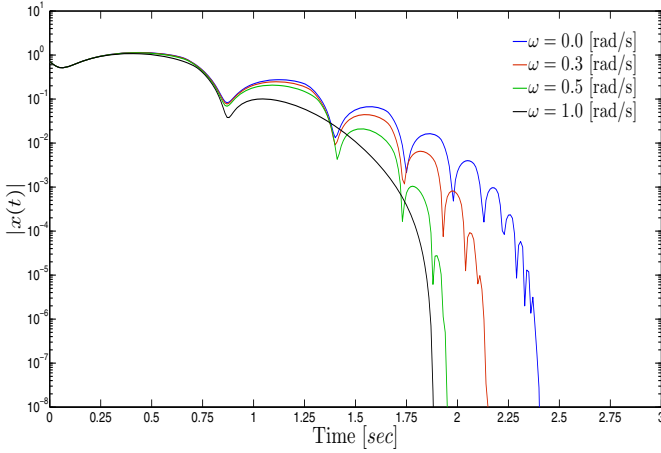
Figure 1. Finite-Time Convergence for Nonlinear Time-Varying System.

inclusion [28]

$$\dot{x}_1(t) \in x_2(t), \tag{8}$$

$$\dot{x}_2(t) \in -k_1 \lceil x_1(t) \rfloor^{\frac{\gamma}{2-\gamma}} - [1,2]k_2 \lceil x_2(t) \rfloor^{\gamma}, \tag{9}$$

System (8)-(9) is r-homogeneous with degree $d = \gamma - 1 < 0$ for $(r_1, r_2) = (2 - \gamma, 1)$, from the differential inclusion point of view given by [29]. Then, it is possible to demonstrate that such a system is GUFTS. Consider the following function

$$V(x_1(t), x_2(t)) = \frac{2-\gamma}{2} |x_1(t)|^{\frac{2}{2-\gamma}} + \frac{1}{2k_1} |x_2(t)|^2.$$

Such a function $V(x_1(t), x_2(t))$ is positive definite, radially unbounded, and continuously differentiable with $\dot{V}(x_1(t), x_2(\tau)) \leq -2\frac{k_2}{k_1} |x_2(t)|^{\gamma+1} = -W(x_1(t), x_2(t))$, negative semi-definite. Note that the limit $W(x_1(t), x_2(t)) \to 0$ implies that $x_2(t)$ approaches to the set $E = \{(x_1, x_2) \in \mathbb{R}^2 \,|\, W(x_1(t), x_2(t)) = 0\}$ as $t \to \infty$. Therefore, $W(x_1(t), x_2(t)) = 0$ implies that $x_2(t) = 0$ and thus $x_1(t) = cte$. However, from (9), it is given that $\dot{x}_2(t) \in -k_1 \lceil x_1(t) \rfloor^{\frac{\gamma}{2-\gamma}}$, which implies also that $x_1(t) = 0$. Thus, the set E only contains the trivial solution, and GAUS is concluded [30]. Finally, from statements given by Corollary 4.3 in [29] and the equivalence of trajectories [28] between system (6)-(7) and (8)-(9), it is obtained that all the statements given by Theorem 1 are satisfied. Therefore, system (6)-(7) is GUFTS.

With the purpose of showing that the rate of convergence can be also improved increasing the frequency; some simulations for $\omega \in [0, 1]$ have been done in MATLAB Simulink environment, with Euler discretization method and sampling time equal to $0.01[sec]$. The simulation results, confirming previously given statements, are presented in Fig. 1. It is easy to see that increasing the frequency an improvement in the rate of convergence is obtained without the need to increase the gains k_1, k_2.

V. APPLICATION TO ADAPTIVE ESTIMATION

In this section the previously proposed results will be applied to analyze the convergence of the error dynamics given by a nonlinear estimation algorithm.

A. Problem statement

Consider the following time-varying system:

$$\frac{dx(t)}{dt} = \Gamma^T(t)\theta,$$

where $\Gamma : \mathbb{R} \to \mathbb{R}^m$ is a continuous function of time, and $\theta \in \mathbb{R}^m$. The regressor vector $\Gamma(t)$ is known and bounded, i.e. $\|\Gamma(t)\| \leq \Gamma_+$, satisfying the well-known persistence of excitation condition [19], and θ is the unknown parameter vector. It is assumed that $x(t)$ is available for measurements, and in order to estimate the vector θ the following nonlinear algorithm can be introduced

$$\dot{\hat{x}}(t) = -k_1 \lceil \hat{x}(t) - x(t) \rfloor^{\gamma} + \Gamma^T(t)\hat{\theta}(t), \quad \hat{x}(0) = 0,$$

$$\dot{\hat{\theta}}(t) = -k_2 \lceil \hat{x}(t) - x(t) \rfloor^{2\gamma-1} \Gamma(t), \quad \hat{\theta}(0) = [0 \ldots 0]^T,$$

where $k_1, k_2 > 0$ are positive gains, and the parameter $\gamma \in (0.5, 1)$ (for the case $\gamma = 1$ the estimator reduces to the well known linear adaptive observer [19]). Let us define the errors $\tilde{x}(t) = \hat{x}(t) - x(t)$ and $\tilde{\theta}(t) = \hat{\theta}(t) - \theta$. Hence, the error dynamics is given by

$$\dot{\tilde{x}}(t) = -k_1 \lceil \tilde{x}(t) \rfloor^{\gamma} + \Gamma^T(t)\tilde{\theta}(t), \tag{10}$$

$$\dot{\tilde{\theta}}(t) = -k_2 \lceil \tilde{x}(t) \rfloor^{2\gamma-1} \Gamma(t). \tag{11}$$

It is clear that the system (10)–(11) has the form (1), and it is also possible to express it in the form (3)

$$\dot{\tilde{x}}(t) = -k_1 \lceil \tilde{x}(t) \rfloor^{\gamma} + \Gamma^T(\omega t)\tilde{\theta}(t), \tag{12}$$

$$\dot{\tilde{\theta}}(t) = -k_2 \lceil \tilde{x}(t) \rfloor^{2\gamma-1} \Gamma(\omega t). \tag{13}$$

System (10)–(11) is r-homogeneous with degree $d = \gamma - 1$ for $(r_1, r_2, \ldots, r_{m+1}) = (1, \gamma, \ldots, \gamma)$. Note that $d < 0$ for all $\gamma \in (0.5, 1)$. It is possible to demonstrate that the system (10)–(11) is GUAS. Firstly, note that it is possible to rewrite system (10)–(11) as follows:

$$\dot{\xi}(t) = A(t)\xi(t) + f(t, \tilde{x}(t)),$$

where $\xi(t) = \begin{bmatrix} \tilde{x}(t) & \tilde{\theta}(t)^T \end{bmatrix}^T \in \mathbb{R}^{1+m}$, $A(t)$ is a bounded and continuous matrix, for almost all $t \geq 0$, that is given by

$$A(t) = \begin{bmatrix} -k_1 & \Gamma^T(t) \\ -k_2\Gamma(t) & 0 \end{bmatrix},$$

and the nonlinear term $f :\to \mathbb{R}^2 \to \mathbb{R}^{1+m}$ is given as follows:

$$f(t, \tilde{x}(t)) = \begin{bmatrix} -k_1 \left(\lceil \tilde{x}(t) \rfloor^{\gamma} - \tilde{x}(t) \right) \\ -k_2 \left(\lceil \tilde{x}(t) \rfloor^{2\gamma-1} - \tilde{x}(t) \right) \Gamma(t) \end{bmatrix}.$$

Define

$$W_2(t, \xi(t)) = \xi^T(t)P(t)\xi(t),$$

where $P(t)$ is a continuously differentiable, symmetric, bounded, and positive definite matrix, i.e. $0 < c_1 I \leq P(t) \leq c_2 I, \quad \forall t \geq 0$, which satisfies the following matrix differential equation:

$$\dot{P}(t) = -P(t)A(t) - A^T(t)P(t) - Q(t), \quad \forall t \geq 0,$$

where $Q(t) \geq c_3 I > 0, \quad \forall t \geq 0$, is a continuous, symmetric, and positive definite matrix. The existence of $P(t)$ and $Q(t)$ follows from the persistence of excitation condition of $\Gamma(t)$ (see, for example, [18], [31], [32]). Define also

$$W_1(\tilde{x}(t), \tilde{\theta}(t)) = \frac{\varsigma_1}{p} V_1^p(\tilde{x}(t), \tilde{\theta}(t)) + \frac{\varsigma_2}{q} V_1^q(\tilde{x}(t), \tilde{\theta}(t)),$$

with

$$V_1(\tilde{x}(t), \tilde{\theta}(t)) = \frac{1}{2\gamma} |\tilde{x}(t)|^{2\gamma} + \frac{1}{2k_2} \tilde{\theta}(t)^T \tilde{\theta}(t),$$

$$\varsigma_1 = 2k_1 \mu c_2 (2\gamma)^{p-1}, \varsigma_2 = \frac{2\mu c_2 \left(k_1^2 + k_2^2 \Gamma_+^2\right) (2\gamma)^{q-1}}{k_1},$$

for some positive constants p, q, μ. Introduce the following Lyapunov function candidate:

$$W(t, \tilde{x}(t), \tilde{\theta}(t)) = W_1(\tilde{x}(t), \tilde{\theta}(t)) + W_2(t, \xi(t)). \tag{14}$$

Its derivative along the trajectories of the system (10)–(11) is given by

$$\dot{W}(t, \tilde{x}(t), \tilde{\theta}(t)) = \left(\varsigma_1 V_1^{p-1}(\tilde{x}(t), \tilde{\theta}(t))\right.$$
$$\left. + \varsigma_2 V_1^{q-1}(\tilde{x}(t), \tilde{\theta}(t))\right) \dot{V}_1(\tilde{x}(t), \tilde{\theta}(t))$$
$$+ \xi^T(t) \left[\dot{P}(t) + P(t)A(t) + A^T(t)P(t)\right] \xi(t)$$
$$+ 2\xi^T(t)P(t)f(\tilde{x}(t), t).$$

Using the fact that $\dot{V}_1(\tilde{x}(t), \tilde{\theta}(t)) = -k_1 |\tilde{x}(t)|^{3\gamma-1}$, $2X^T P Y \leq \frac{1}{\mu} X^T P X + \mu Y^T P Y$ for any $Y \in \mathbb{R}^{1+m}$ and $\mu > 0$, it follows that

$$\dot{W}(t, \tilde{x}(t), \tilde{\theta}(t)) \leq -k_1 \left(\varsigma_1 V_1^{p-1}(\tilde{x}(t), \tilde{\theta}(t)) + \right.$$
$$\left. \varsigma_2 V_1^{q-1}(\tilde{x}(t), \tilde{\theta}(t))\right) \left(|\tilde{x}(t)|^{3\gamma-1}\right) -$$
$$\xi^T(t)Q(t)\xi(t) + \frac{1}{\mu}\xi^T(t)P(t)\xi(t) + \mu f^T(\tilde{x}(t), t)P(t)f(\tilde{x}(t), t),$$

$$\dot{W}(t, \tilde{x}(t), \tilde{\theta}(t)) \leq -\eta \|\xi\|^2 + \mu c_2 \|f\|^2$$
$$-k_1 \left(\varsigma_1 V_1^{p-1}(\tilde{x}(t), \tilde{\theta}(t)) + \varsigma_2 V_1^{q-1}(\tilde{x}(t), \tilde{\theta}(t))\right) |\tilde{x}(t)|^{3\gamma-1},$$

where $\eta = c_3 - \frac{c_2}{\mu}$. Note that $V_1(\tilde{x}(t), \tilde{\theta}(t)) \geq \frac{1}{2\gamma} |\tilde{x}(t)|^{2\gamma}$, and the term $\mu c_2 \|f\|^2$ can be bounded as follows:

$$\mu c_2 \|f\|^2 \leq \mu c_2 \left(2k_1^2 \left(|\tilde{x}(t)|^{2\gamma} + |\tilde{x}(t)|^2\right) + \right.$$
$$\left. + k_2^2 \Gamma_+^2 \left(|\tilde{x}(t)|^{4\gamma-2} + |\tilde{x}(t)|^2\right)\right),$$

therefore

$$\dot{W}(t, \tilde{x}(t), \tilde{\theta}(t)) \leq -\eta \|\xi\|^2$$
$$+ 2\mu c_2 k_1^2 |\tilde{x}(t)|^{2\gamma} - \frac{k_1 \varsigma_1}{(2\gamma)^{p-1}} |\tilde{x}(t)|^{\gamma(2p+1)-1}$$
$$+ 2\mu c_2 k_1^2 |\tilde{x}(t)|^2 - \frac{k_1 \varsigma_2}{(2\gamma)^{q-1}} |\tilde{x}(t)|^{\gamma(2q+1)-1}$$
$$+ \mu c_2 k_2^2 \Gamma_+^2 \left(|\tilde{x}(t)|^{4\gamma-2} + |\tilde{x}(t)|^2\right).$$

Then, for all $|\tilde{x}(t)| > 1$, it is given that $|\tilde{x}(t)|^{4\gamma-2} + |\tilde{x}(t)|^2 \leq 2 |\tilde{x}(t)|^2$, and

$$\dot{W}(t, \tilde{x}(t), \tilde{\theta}(t)) \leq -\eta \|\xi\|^2$$
$$+ 2\mu c_2 k_1^2 |\tilde{x}(t)|^{2\gamma} - \frac{k_1 \varsigma_1}{(2\gamma)^{p-1}} |\tilde{x}(t)|^{\gamma(2p+1)-1}$$
$$+ 2\mu c_2 \left(k_1^2 + k_2^2 \Gamma_+^2\right) |\tilde{x}(t)|^2 - \frac{k_1 \varsigma_2}{(2\gamma)^{q-1}} |\tilde{x}(t)|^{\gamma(2q+1)-1}.$$

Thus, the parameters p and q will be designed such that every positive term, in $\dot{W}(t, \tilde{x}(t), \tilde{\theta}(t))$, is compensated, *i.e.* such that

$$|\tilde{x}(t)|^{2\gamma} = |\tilde{x}(t)|^{\gamma(2p+1)-1},$$
$$|\tilde{x}(t)|^2 = |\tilde{x}(t)|^{\gamma(2q+1)-1}.$$

Therefore, in order to ensure that $\dot{W}(t, \tilde{x}(t), \tilde{\theta}(t)) \leq -\eta \|\xi\|^2$, for all $|\tilde{x}(t)| > 1$, the following constraints are obtained

$$p = \frac{\gamma+1}{2\gamma}, \quad q = \frac{3-\gamma}{2\gamma}.$$

Note that when $\gamma \in (0.5, 1)$, it is obtained that $p \in (1, 1.5)$ and $q \in (1, 2.5)$. Thus, $W(t, \tilde{x}(t), \tilde{\theta}(t))$ is positive definite, radially unbounded, continuous in $\mathbb{R}^{1+m}$ and continuously differentiable for all $|\tilde{x}(t)| > 1$. For the case $|\tilde{x}(t)| \leq 1$, it is given that $|\tilde{x}(t)|^{4\gamma-2} + |\tilde{x}(t)|^2 \leq 2$, and

$$\dot{W}(t, \tilde{x}(t), \tilde{\theta}(t)) \leq -\eta \|\xi\|^2 + 2\mu c_2 k_2^2 \Gamma_+^2$$
$$+ 2\mu c_2 k_1^2 |\tilde{x}(t)|^{2\gamma} - \frac{k_1 \varsigma_1}{(2\gamma)^{p-1}} |\tilde{x}(t)|^{\gamma(2p+1)-1}$$
$$+ 2\mu c_2 k_1^2 |\tilde{x}(t)|^2 - \frac{k_1 \varsigma_2}{(2\gamma)^{q-1}} |\tilde{x}(t)|^{\gamma(2q+1)-1}.$$

Since the positive terms $|\tilde{x}(t)|^{2\gamma}$ and $|\tilde{x}(t)|^2$, can be compensated by the negative terms $|\tilde{x}(t)|^{\gamma(2p+1)-1}$ and $|\tilde{x}(t)|^{\gamma(2q+1)-1}$, for all $\tilde{x}$; it is obtained that $\forall |\tilde{x}(t)| \leq 1$

$$\dot{W}(t, \tilde{x}(t), \tilde{\theta}(t)) \leq -\eta \|\xi\|^2 + 2\mu c_2 k_2^2 \Gamma_+^2.$$

It is possible to demonstrate that $W(\tilde{x}(t), \tilde{\theta}(t)) \geq \frac{\varsigma_1}{p} \alpha^p \|\xi\|^{2p} + \frac{\varsigma_2}{q} \alpha^q \|\xi\|^{2q} + c_1 \|\xi\|^2$, with $\alpha = \min(\frac{1}{2\gamma}, \frac{1}{2k_2})$; and since $p \in (1, 1.5)$ and $q \in (1, 2.5)$, it is given that $W(\tilde{x}(t), \tilde{\theta}(t)) \geq \phi \|\xi\|^5, \forall |\tilde{x}(t)| \leq 1$, with $\phi \geq \frac{\varsigma_1}{p} \alpha^p + \frac{\varsigma_2}{q} \alpha^q + c_1$. Therefore, it follows that $\forall |\tilde{x}(t)| \leq 1$

$$\dot{W}(t, \tilde{x}(t), \tilde{\theta}(t)) \leq -\frac{\eta}{\phi^{\frac{2}{5}}} W^{\frac{2}{5}}(\tilde{x}(t), \tilde{\theta}(t)) + 2\mu c_2 k_2^2 \Gamma_+^2,$$

and US is obtained. Thus, based on the statements given by Corollary 5, the following result has been proven.

Theorem 2. *Let the vector $\Gamma(t)$ be persistently exciting. Then, the system (10)–(11) is GUAS at the origin.*

Since system (10)–(11) is also homogeneous, the rate of convergence for its modified version (12)–(13) can be evaluated using Lemma 1 from the convergence of the original system (10)–(11). The modification (12)–(13) corresponds to a frequency change in the regressor vector, that is a usual instrument in the adaptive estimation. According to the results of Proposition 2 and Lemma 1, for the linear estimator with

$\gamma = 1$ the speed of error convergence should not be modified, while for the nonlinear observer (12)–(13) with $\gamma \in (0.5, 1)$ the frequency ω may impact the convergence (note that the form of the function $\beta \in \mathcal{KL}$, from Lemma 1, is unknown, in general a rescaling by $\omega^{-1/d}$ of the initial conditions may cancel the rate improvement by ωt).

B. Example

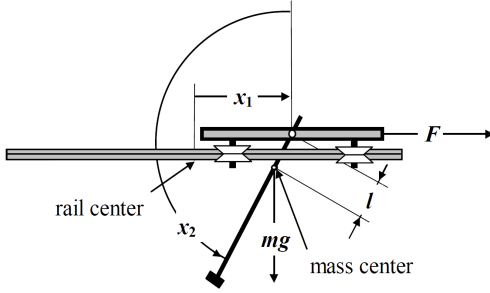Consider the Pendulum-Cart system depicted in Fig. 2. A

Figure 2. Pendulum-Cart System.

pendulum rotates in a vertical plane around an axis located on a cart. The cart can move along a horizontal rail, lying in the plane of rotation. The system state is a vector $x = [x_1, x_2, x_3, x_4]^T$, where x_1 is the cart position, x_2 is the angle between the upward direction and the pendulum, measured counterclockwise ($x_2 = 0$ for the upright position of the pendulum), x_3 is the cart velocity, and x_4 is the pendulum angular velocity. A control force F, parallel to the rail, is applied to the cart. It is produced by a DC flat motor controlled by a pulse-width-modulation (PWM) voltage signal u, and $F = p_1 u$. The system control u takes values in the interval $[-0.5, 0.5]$. The total mass of the pendulum and cart is denoted by m. The distance from the axis of rotation of the pendulum to the center of mass of the system is l. The moment of inertia of the pendulum with respect to its axis on the cart is represented by J_p. The cart friction is compound of two forces: the static and the viscous friction proportional to the cart velocity, $f_c x_3$ and $f_s \text{sign}(x_3)$, respectively. There is also a friction torque in the angular motion of the pendulum, proportional to the angular velocity, $f_p x_4$. The system dynamics is described by the following equations

$$\dot{x}_1(t) = x_3(t), \tag{15}$$
$$\dot{x}_2(t) = x_4(t), \tag{16}$$
$$\dot{x}_3(t) = \frac{a_1 w_1(x(t), u(t)) + a_2(x_2(t)) w_2(x(t))}{d(x_2(t))}, \tag{17}$$
$$\dot{x}_4(t) = \frac{a_2(x_2(t)) w_1(x(t), u(t)) + w_2(x(t))}{d(x_2(t))}, \tag{18}$$
$$y(t) = [x_1(t)\ x_2(t)]^T, \tag{19}$$

where

$$m = m_c + m_p,\ a_1 = l^2 + \frac{J}{m},\ a_2(x_2(t)) = l\cos(x_2(t)),$$
$$w_1(x(t), u(t)) = p_1 u(t) - f_s \text{sign}(x_3(t)) - f_c x_3(t)$$
$$\quad - m l x_4^2(t)\sin(x_2(t)),$$
$$w_2(x(t)) = m l g \sin(x_2(t)), -f_p x_4(t),$$
$$d(x_2(t)) = J + m l^2 \sin^2(x_2(t)).$$

The parameters of the model are given in Table I. Let us

Table I
PARAMETERS OF THE PENDULUM-CART SYSTEM.

Name	Description	Unit
m_c	Car mass	$0.5723[kg]$
m_p	Pendulum mass	$0.12[kg]$
l	Distance from the rotation axis to the center of mass	$0.0196[m]$
f_c	Dynamic friction coefficient	$0.5[Ns/m]$
f_s	Dry friction coefficient	$1.1976[kg/m]$
f_p	Rotational friction coefficient	$0.1354[Nms/rad]$
J	Moment of inertia of the pendulum	$0.0039[kgm^2]$
g	Gravity	$9.81[m/s^2]$
p_1	Control force to PWM signal ratio	$9.4[N]$

assume that it is necessary to identify the friction parameters, *i.e.* f_s, f_c, f_p, and the control force to PWM signal radio, *i.e.* p_1, therefore, the dynamics given by (17) can be rewritten as follows

$$\dot{x}_3(t) = \Gamma^T(t)\theta(t) + f(x(t)), \tag{20}$$

where

$$\Gamma^T(t) = \frac{a_1}{d(x_2(t))} \begin{bmatrix} -\text{sign}(x_3(t)) & -x_3(t) & -x_4(t) & u(t) \end{bmatrix},$$
$$\theta^T(t) = \begin{bmatrix} f_s & f_c & f_p & p_1 \end{bmatrix},$$
$$u(t) = 0.4\sin(\omega t) + 0.1\cos(0.1\omega t),$$
$$f(x(t)) = \frac{a_2(x_2(t)) m l g \sin(x_2(t)) - a_1 m l x_4^2(t)\sin(x_2(t))}{d(x_2(t))}.$$

Note that it is enough to consider dynamics (20) in order to identify the corresponding parameters. Assume that the state vector $x(t)$ is measurable. The parameter identification algorithm takes the following form

$$\dot{\hat{x}}_3(t) = -k_1 \lceil \hat{x}_3(t) - x_3(t) \rceil^\gamma + \Gamma^T(t)\hat{\theta}(t) + f(x(t)),$$
$$\dot{\hat{\theta}}(t) = -k_2 \lceil \hat{x}_3(t) - x_3(t) \rceil^{2\gamma - 1}\Gamma(t),$$

where $\hat{x}_3(0) = 0$ and $\hat{\theta}(0) = [0.8\ 0.1\ 0\ 4.5]^T$. Let us consider the case $\gamma = 0.70$ with $k_1 = 35$, $k_2 = 3.5$. Some simulations, for different values of ω, have been done in MATLAB Simulink environment, with Euler discretization method and sampling time equal to $0.01[sec]$. The simulation results, confirming Lemma 1 statements, are presented in Figs. 3-4 in comparison with the linear estimation algorithm for $\gamma = 1$. As a conclusion, increasing the frequency, it is possible to improve the rate of convergence for the nonlinear algorithm in certain limits, while for the linear algorithm there is no significant improvement.

Therefore, application of the nonlinear homogeneous algorithms may serve for improvement of the rate of estimation
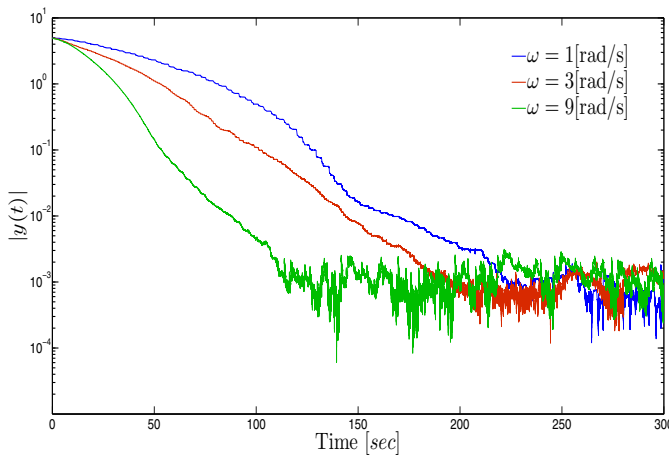
Figure 3. Estimation Error Convergence for Nonlinear Adaptive Observer ($\gamma = 0.70$), $\xi(t) = \begin{bmatrix} \tilde{x}_3(t) & \tilde{\theta}^T(t) \end{bmatrix}^T$.
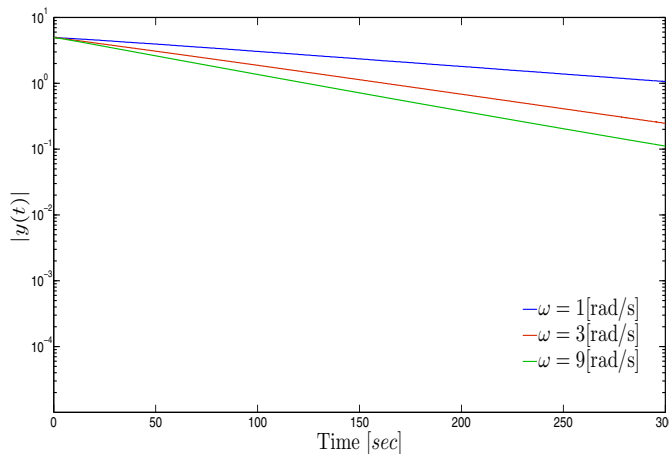
Figure 4. Estimation Error Convergence for Linear Adaptive Observer ($\gamma = 1$), $\xi(t) = \begin{bmatrix} \tilde{x}_3(t) & \tilde{\theta}^T(t) \end{bmatrix}^T$.

depending on the available excitation frequency of the time-varying regressor.

VI. CONCLUSIONS

The homogeneity theory extensions are obtained for time-varying systems. It is shown that for any degree of homogeneity the solutions of a homogeneous system are interrelated subject to the time rescaling. Next, this fact is utilized in order to show that local uniform asymptotic stability of homogeneous systems implies global one, and that for nonlinear homogeneous systems with non-zero degree global asymptotic stability for a parameter endorses this property for an arbitrary value of this parameter. The possibility of finite-time stability in time-varying systems is discussed. Efficiency of the proposed approach is demonstrated for an adaptive estimation problem benchmark. Application of the developed results for analysis and design of control or estimation algorithms in time-varying systems is a direction of future research. Improvement of rate of convergence in the adaptive estimation algorithm by parameter tuning based on the proposed technique is another possible direction of future work.

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